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Mathematics - 1st Term - Part 2

Contents

Chapter 1

Polynomial functions and rational fractions

Polynomial functions are the most simple functions that we can encounter. They are the result of repeated applications of simple arithmetic operations: addition and multiplication.

1.1 Polynomial functions

1.1.1 Definition and properties

Definition 1.1. *A* **polynomial function** *with coefficients in* $\mathbb{K}$ *(*$\mathbb{K} = \mathbb{R}$ *ou* $\mathbb{C}$*) is any function defined on* $\mathbb{R}$ *or* $\mathbb{C}$ *of the form:*

$$P(x) = a_n x^n + \cdots + a_1 x + a_0 \text{ with } \forall\ i \in \{0, 1, \cdots, n\},\ a_i \in \mathbb{K},\ n \in \mathbb{N}.$$

$a_0, a_1, \cdots, a_n$ *are called the coefficients of* P.
We denote as $\mathbb{K}[x]$ *the set of polynomial functions with coefficients in* $\mathbb{K}$.

Definition 1.2. *The* **zero polynomial function** *or* **zero polynomial**, *denoted* $0_{\mathbb{K}[x]}$, *is the polynomial function whose coefficients are all equal to* 0.

Definition 1.3. *Let*

$$P(x) = a_n x^n + \cdots + a_1 x + a_0 \text{ with } \forall\ i \in \{0, 1, \cdots, n\},\ a_i \in \mathbb{K} \text{ and } a_n \neq 0.$$

1. *The* **degree** *of* P *is the biggest power of* x *that appears in* $P(x)$. *It is denoted* $d(P)$ *or* $deg(P)$. *Here* $d(P) = n$ *as* $a_n \neq 0$.

2. *The* **valuation** *of* P *is the smallest power of* x *that appears in* $P(x)$. *It is denoted* $v(P)$ *or* $val(P)$.

3. *The* **constant term** *of* $P(x)$ *is* a_0.

4. P *is a* **monic polynomial function** *if the highest degree coefficient (here,* a_n*) is equal to* 1.

Example 1.

1. *$P(x) = 5x^4 - 4x^2 + 3x + 2$ is a non-monic polynomial function. Its degree is $d(P) = 4$ and its valuation is $v(P) = 0$. Its constant term is 2.*

2. *We consider the polynomial functions $P_1(x) = 4x^3 - 2x$, $P_2(x) = x^6 - 2x^2 - 3$ et $P_3(x) = 4$. Give the degree, valuation, and constant term of these polynomials and specify whether or not they are monic.*

Remark 1. *By convention, $d(0_{\mathbb{K}[x]}) = -\infty$ and $v(0_{\mathbb{K}[x]}) = +\infty$.*
Beware, the degree of $P(x) = a_0 \neq 0$ is 0.

The subset of polynomial functions of highest degree n is denoted $\mathbb{K}_n[x]$.

We now introduce the term **polynomial** as a shortcut for a polynomial function.

Elementary operations

Let A and B in $\mathbb{K}[x]$. We denote as $a_k, k \in \mathbb{N}$ (respectively $b_k, k \in \mathbb{N}$) the coefficients of A (respectively B).

- The **sum** of A and B is a polynomial S whose coefficients $s_k, k \in \mathbb{N}$ are given by: $s_k = a_k + b_k, k \in \mathbb{N}$.
  More generally, for any real λ and μ, $\lambda A + \mu B$ is a polynomial whose coefficients are: $\lambda a_k + \mu b_k, k \in \mathbb{N}$.

  We can check that

  $$\boxed{d(A+B) \leq \max\{d(A),\, d(B)\} \text{ and } v(A+B) \geq \min\{v(A),\, v(B)\}.}$$

- Two polynomials A and B are **equal** if their difference is zero. It is equivalent to their coefficients all being equal.

- The **product** of A and B is a polynomial P whose coefficients denoted $p_k, k \in \mathbb{N}$ are given by the formula : $p_k = \displaystyle\sum_{i=0}^{k} a_i\, b_{k-i}$.

  We can check that

  $$\boxed{d(A\,B) = d(A) + d(B) \text{ and } v(A\,B) = v(A) + v(B).}$$

Remark 2. *Using the convention on the degree and valuation of the zero polynomial, the formulae for the degree and valuation of the sum and product remain true when at least one of the polynomials is a zero polynomial.*

Example 1.

1. *The sum of the polynomials* $A(x) = -x^3 + x^2 + 3x$ *and* $B(x) = x^3 + 2x + 1$ *is the polynomial* $x^2 + 5x + 1$. *Their product is* $P(x) = -x^6 + x^5 + x^4 + x^3 + 7x^2 + 3x$.
2. *Let* $A_1(x) = 2x^2 - 3x + 2$ *and* $B_1(x) = -x^2 - 3x + 3$. *Compute* $A_1 + 2B_1$, $3A_1 - 2B_1$ *and* $A_1\, B_1$. *Compare* $d(A_1 + 2B_1)$ *(respectively* $d(3A_1 - 2B_1)$ *and* $d(A_1\, B_1)$*) with* $d(A_1)$ *and* $d(B_1)$. *Repeat the question for the valuations.*

- In general, the **quotient** of two polynomials is not a polynomial. The fraction $1/x$ is an example of this. However, just as we do for whole numbers, we can define a polynomial division with remainder. In the following we discuss the most useful form of such polynomial division: Euclidean division.

1.1.2 Polynomial Euclidean division

Theorem 1.4. *Let* A *and* B *be two elements of* $\mathbb{K}[x]$, *with* $B \neq 0_{\mathbb{K}[x]}$. *There exists a unique couple* $(Q,\, R)$ *of elements such that:*

$$\boxed{\begin{cases} (i) & A = B\,Q + R, \\ (ii) & d(R) < d(B). \end{cases}}$$

Proof
Uniqueness.

Existence.

- If $d(A) < d(B)$, we can simply set $Q = 0$ and $R = A$. We then obtain $A = BQ + R$ with $d(R) < d(B)$.
- If $d(A) \geq d(B)$, we reason by induction on the degree n of A.

Induction hypothesis : For $n \in \mathbb{N}$, let $\mathcal{P}(n)$ be the statement : "For any A whose degree is inferior or equal to n, for any $B \neq 0_{\mathbb{K}[x]}$, there exists (Q, R) such that (i) and (ii) are satisfied."

Base case : If $d(A) = 0$, then $A = a_0$ with $a_0 \in \mathbb{K}^*$.
As $B \neq 0_{\mathbb{K}[x]}$ and $0 = d(A) \geq d(B)$ we have that $d(B) = 0$ so $B = b_0 \in \mathbb{K}^*$. We then set $Q = a_0/b_0$ and $R = 0$. Conditions (i) and (ii) are therefore verified .

Inductive step : Assume the statement holds for some n. It must then be shown that the statement holds for $n + 1$. We consider $A = a_{n+1}x^{n+1} + \ldots + a_0$ a polynomial of degree $n + 1$ ($a_{n+1} \neq 0$). We also denote $B = b_p x^p + \ldots + b_0$ a polynomial of degree p ($b_p \neq 0$) with $p \leq n + 1$. We define

$$Q_1 = \frac{a_{n+1}}{b_p} x^{n+1-p} \quad \text{and} \quad A_1 = A - BQ_1.$$

By construction, the degree of A_1 is smaller or equal to n. Thus, using the induction hypothesis, there exists a couple (Q_2, R_2) such that $A_1 = BQ_2 + R_2$ with $d(R_2) < d(B)$. Consequently, $A = BQ + R$ with $Q = Q_1 + Q_2$ and $R = R_2$ thereby showing that $P(n + 1)$ holds. Since both the base case and the inductive step have been performed, by mathematical induction the statement $P(n)$ holds for all natural numbers n and the theorem holds.

$\square$

Definition 1.5. *Performing* **Euclidean division** *of a polynomial A by a polynomial B means finding the unique couple of polynomials (Q, R) such that $A = B\,Q + R$ with $d(R) < d(B)$.*
A is called the **dividend**, *B is called the* **divisor**, *Q is called the* **quotient** *and R is called the* **remainder** *of the* **Euclidean division** *of A by B. When $R = 0_{\mathbb{K}[x]}$, B is said to* **divide** *A.*

We will now illustrate through some examples how to perform this Euclidean division in practice.

Example 2.

1. *When $d(A) < d(B)$ no computation is necessary. Let us study for example the Euclidean division of $A(x) = 3x + 1$ by $B(x) = x^2 + 1$.*
   We have that $A = 0_{\mathbb{K}[x]}B + A$ and through uniqueness of Euclidean division we obtain that $Q = 0_{\mathbb{K}[x]}$ et $R = A$.

2. *We now consider the case where $d(A) = d(B)$. Let us study for example the Euclidean division of $A(x) = -x^3 + x^2 + 3x$ by $B(x) = x^3 + 2x + 1$.*
   Here, $d(B) = d(A)$ and $A(x) + B(x) = x^2 + 5x + 1$: We immediately have that $A(x) = (-1) \times B(x) + (x^2 + 5x + 1)$. The quotient is the constant polynomial $Q(x) = -1$, the remainder is $R(x) = x^2 + 5x + 1$.

3. *Finally, when $d(A) > d(B)$, we have to perform long division in a similar way to when we compute Euclidean division for whole numbers. Let us study for example the Euclidean division of $A(x) = x^4 - 3x + 1$ by $B(x) = x^2 + 1$.*

$$
\begin{array}{rrrrrr|rr}
 & x^4 & & & -3x & +1 & x^2 & +1 \\ \cline{7-8}
- & (x^4 & & +x^2) & & & x^2 & -1 \\ \cline{1-6}
 & & & -x^2 & -3x & +1 & & \\
 & & - & (-x^2 & & -1) & & \\ \cline{3-6}
 & & & & -3x & +2 & &
\end{array}
$$

We thus obtain $x^4 - 3x + 1 = (x^2 + 1)(x^2 - 1) + (-3x + 2)$.

Example 3. *Compute the following Euclidean divisions:*

1. *$A_1(x) = 2x^2 - 3x + 2$ by $B_1(x) = -x^2 - 3x + 3$,*

2. *$A_2(x) = x^4 - 3x^2 + 1$ by $B_2(x) = x^2 + 1$,*

3. *$A_3(x) = x + 1$ by $B_3(x) = x^2 - 3$.*

A specific method, called Horner's method, can be used to compute Euclidean division by a monic polynomial of degree 1 (of the form $x - \alpha$).

Example 4 (Horner's method)**.** *We wish to perform Euclidean division of* $x^4 + 3x^3 + x + 1$ *by* $x - 2$.
In the following table, the entries in the first row are the coefficients of the dividend. α *is written down in the second row of the left hand column. The first coefficient is immediately copied in the bottom row. We then iterate the following operations:*

- *we multiply the last term written in the third row by* α *and we write the result in the second row of the following column,*
- *we sum the last term written in the second row with the term written just above it and we write the result in the third row.*

$$\begin{array}{c|ccc||cc}
 & 1 & 3 & 0 & 1 & 1 \\
2 & & 2 & 10 & 20 & 42 \\
\hline
 & 1 & 5 & 10 & 21 & 43
\end{array}$$

The entries to the left of the double bar in the last row are the ordered coefficients of the quotient. The term to the right of the double bar in the last row is the rest. So we obtain

$$x^4 + 3x^3 + x + 1 = (x^3 + 5x^2 + 10x + 21)(x - 2) + 43.$$

Using Horner's method leads to the same computations as long division. It is just a quicker and more condensed way of writing it out.

Remark 3. *To compute the value of a polynomial* P *for* $x = \alpha$*, we can perform Euclidean division of* P *by* $x - \alpha$*. Indeed, we then obtain*

$$P(x) = Q(x)(x - \alpha) + R,$$

and immediately deduce $P(\alpha) = R$.
For example to compute the value of $P(x) = x^5 + 3x^3 + 2x^2 + 1$ *for* $x = 3$*, we can perform Euclidean division by* $x - 3$ *using Horner's method*

$$\begin{array}{c|cccc||cc}
 & 1 & 0 & 3 & 2 & 0 & 1 \\
3 & & 3 & 9 & 36 & 114 & 342 \\
\hline
 & 1 & 3 & 12 & 38 & 114 & 343
\end{array}$$

Thus $P(x) = (x - 3)(x^4 + 3x^3 + 12x^2 + 38x + 114) + 343$ *and so* $P(3) = 343$.

1.1.3 Polynomial factorization

Roots of polynomials

Definition 1.6. *The number $\alpha \in \mathbb{K}$ is said to be a* **root** *of $A \in \mathbb{K}[x]$ if $A(\alpha) = 0$.*

Theorem 1.7. *The number $\alpha \in \mathbb{K}$ is a root of $A \in \mathbb{K}[x]$ if and only if $x - \alpha$ divides A.*

Proof

$\square$

We can explicitly compute the roots of a polynomial of degree 1 or 2. When it comes to polynomials with a degree greater or equal to 3, one possible method is to first find one or several trivial roots. We then factorize the polynomial to work with lower degree polynomials.

Example 5. *One root of the polynomial $x^3 - 6x^2 + 12x - 8$ is 2. Euclidean division of $x^3 - 6x^2 + 12x - 8$ by $x - 2$ can be computed:*

$$
\begin{array}{rrrrr|rrr}
 & x^3 & -6x^2 & +12x & -8 & & x & -2 \\ \cline{6-8}
- & (x^3 & -2x^2) & & & x^2 & -4x & +4 \\ \cline{2-5}
 & & -4x^2 & +12x & -8 & & & \\
 & - & (-4x^2 & +8x) & & & & \\ \cline{3-5}
 & & & 4x & -8 & & & \\
 & & - & (4x & -8) & & & \\ \cline{4-5}
 & & & & 0 & & &
\end{array}
$$

As in addition, $x^2 - 4x + 4 = (x - 2)^2$, we finally obtain $x^3 - 6x^2 + 12x - 8 = (x - 2)^3$.

Multiple roots

Definition 1.8. *A root α is said to be of* **multiplicity** *k for $P(x)$, if one of the following and equivalent statements is verified :*

i) $(x-\alpha)^k$ *divides P with* $P(x) = (x-\alpha)^k\, Q(x)$, *and* $Q(\alpha) \neq 0$.

ii) $(x-\alpha)^k$ *divides P and* $(x-\alpha)^{k+1}$ *does not divide P.*

When $k = 1$, a root is said to be **simple**.

Multiple roots can also be characterised through the derivatives of P. To show this definition, we first need the following preliminary result.

Lemma 1. *Let $\alpha \in \mathbb{K}$, $k \in \mathbb{N}$ and $\ell \in \mathbb{N}$ such that $0 \leq \ell \leq k$. Let P and Q be two polynomials with coefficients in $\mathbb{K}$. If $P(x) = (x-\alpha)^k Q(x)$ then the ℓ^{th} derivative of P can be written as*

$$P^{(\ell)}(x) = \sum_{j=0}^{\ell} \mathcal{C}_\ell^j [k(k-1)\cdots(k-j+1)](x-\alpha)^{k-j} Q^{(\ell-j)}(x).$$

Proof This result is a consequence of the Leibniz formula (*cf* Chapter 2, Theorem 2.35). It can also be shown by induction. □

Let us now give the other characterisation of multiple roots.

Proposition 1.9. *Let $P \in \mathbb{K}[x]$ and $\alpha \in \mathbb{K}$. Then α is a root of multiplicity k of P if and only if $P(\alpha) = P'(\alpha) = \ldots = P^{(k-1)}(\alpha) = 0$ et $P^{(k)}(\alpha) \neq 0$.*

Proof If α is a root of multiplicity k of P then $P(x) = (x-\alpha)^k Q(x)$ avec $Q(\alpha) \neq 0$. According to Lemma 1, for all $\ell \in \mathbb{N}$ such that $0 \leq \ell \leq k$,

$$P^{(\ell)}(x) = \sum_{j=0}^{\ell} \mathcal{C}_\ell^j [k(k-1)\cdots(k-j+1)](x-\alpha)^{k-j} Q^{(\ell-j)}(x),$$

so for $0 \leq \ell \leq k-1$,

$$P^{(\ell)}(\alpha) = \sum_{j=0}^{\ell} \mathcal{C}_\ell^j [k(k-1)\cdots(k-j+1)](\alpha-\alpha)^{k-j} Q^{(\ell-j)}(\alpha) = 0,$$

and for the k^{th} derivative,

$$P^{(k)}(\alpha) = \sum_{j=0}^{k} \mathcal{C}_k^j [k(k-1)\cdots(k-j+1)](\alpha-\alpha)^{k-j} Q^{(k-j)}(\alpha) = k! Q(\alpha) \neq 0.$$

Conversely, we show the following property by induction over $k \in \mathbb{N}^*$:

$$\mathcal{S}(k) : \text{“ if } P(\alpha) = P'(\alpha) = \ldots = P^{(k-1)}(\alpha) = 0 \text{ then } (x-\alpha)^k \text{ divides } P\text{”}.$$

Base case : for $k=1$, $P(\alpha)=0 \Longrightarrow (x-\alpha)$ divides P so $\mathcal{S}(1)$ is true.

Induction hypothesis : Suppose that there exists $k \in \mathbb{N}^*$ such that $\mathcal{S}(k)$ is true.

Inductive step : Suppose that $P(\alpha)=P'(\alpha)=\ldots=P^{(k-1)}(\alpha)=P^{(k)}(\alpha)=0$. In particular $P(\alpha)=P'(\alpha)=\ldots=P^{(k-1)}(\alpha)=0$ so through $\mathcal{S}(k)$, there exists a polynomial Q_1 such that
$P(x)=(x-\alpha)^k Q_1(x)$ and through the previous computation, $P^{(k)}(\alpha)=k!Q_1(\alpha)$. As $P^{(k)}(\alpha)=0$, it holds that $Q_1(\alpha)=0$. As a consequence there exists a polynomial Q such that $Q_1(x)=(x-\alpha)Q(x)$ and so such that $P(x)=(x-\alpha)^{k+1}Q(x)$. Therefore, $\mathcal{S}(k+1)$ holds and through the induction principle, for any $k \in \mathbb{N}^*$,

$$P(\alpha)=P'(\alpha)=\ldots=P^{(k-1)}(\alpha)=0 \Longrightarrow \text{ there exists } Q \in \mathbb{R}[x] \text{ such that } (x-\alpha)^k Q(x).$$

Furthermore as $P^{(k)}(\alpha)=k!Q(\alpha)\neq 0$, we obtain that $Q(\alpha)\neq 0$ so α is a root of multiplicity k of P. □

Example 6.

1) 2 is a triple root ($k=3$) of $x^3-6x^2+12x-8$.

2) Find the roots of $Q(x)=x^3-7x^2+15x-9$. Specify their multiplicity.

Definition 1.10. *$P \in \mathbb{K}[x]$ is said to* **split** *over $\mathbb{K}$ if there exists λ, $\alpha_1,\ldots,\alpha_l \in \mathbb{K}$ and k_1, $k_l \in \mathbb{N}$ such that*

$$P(x)=\lambda(x-\alpha_1)^{k_1}\,\ldots\,(x-\alpha_l)^{k_l}.$$

We deduce that a n^{th} degree polynomial that splits over $\mathbb{K}$ has exactly n roots in $\mathbb{K}$ (counted with their multiplicity).

Sum and product of the roots of a polynomial

Theorem 1.11. *Let $(a,b,c)\in\mathbb{C}^3$ with $a\neq 0$. The roots r_1 and r_2 of the trinomial $a\,x^2+b\,x+c$, verify the following statements :*

$$r_1+r_2=-\frac{b}{a} \quad \textit{and} \quad r_1\,r_2=\frac{c}{a}. \tag{1.1}$$

Proof We denote $\Delta=b^2-4\,a\,c$. According to Proposition **??** there exists $\delta\in\mathbb{C}$ such that $\delta^2=\Delta$ and the roots of the equation are $r_1=\frac{-b+\delta}{2a}$, $r_2=\frac{-b-\delta}{2a}$ (these roots are equal if $\Delta=0$) and the statements are verified. □

Corollary 1. *Let $(a,b,c)\in\mathbb{R}^3$ with $a\in\mathbb{R}^*$.*

1. *If r_1 and r_2 are the real roots of the trinomial $a\,x^2+b\,x+c$, then by denoting $S=r_1+r_2$ and $P=r_1r_2$, then r_1 and r_2 are the roots of the polynomials x^2-Sx+P and $x^2-Sx+P=(x-r_1)(x-r_2)$.*

2. *Moreover, if* $\Delta = b^2 - 4\,a\,c < 0$ *then* $r_1 = \alpha \in \mathbb{C}$ *and* $r_2 = \overline{\alpha}$*, so* $S = \alpha + \bar{\alpha} = 2\ Re(\alpha)$ *and* $P = \alpha\bar{\alpha} = |\alpha|^2$*. So* α *and* $\bar{\alpha}$ *are roots of* $x^2 - 2\ Re(\alpha)x + |\alpha|^2$ *and we can write* $(x-\alpha)(x-\bar{\alpha}) = x^2 - 2\ Re(\alpha)x + |\alpha|^2$.

The statements (1.1) generalise to polynomials of higher degrees.

Theorem 1.12. *Let* $P(x) = a_n\, x^n + \ldots + a_0$ *be a polynomial that splits over* $\mathbb{K}$*. We denote* r_i, $1 \le i \le n$ *the roots of* P*, counted with their multiplicity. Then,*

i) $\displaystyle\sum_{i=1}^{n} r_i = -\frac{a_{n-1}}{a_n}$

ii) $\displaystyle\prod_{i=1}^{n} r_i = (-1)^n\,\frac{a_0}{a_n}$.

Proof

1.1.4 Number of roots and de d'Alembert's theorem

Theorem 1.13. *A n^{th} degree polynomial with $n \geq 0$ has at most n roots counted with their multiplicity.*

Proof

$\square$

Corollary 2. *If the polynomial $P \in \mathbb{K}_n[x]$ has $l > n$ roots, then $P = 0_{\mathbb{K}[x]}$.*

Proof

$\square$

Corollary 3. *Two n^{th} degree polynomials A and B that have equal values at $n+1$ different points are equal.*

Proof Exercice. $\square$

We now know that a n^{th} degree polynomial has at most n roots in $\mathbb{R}$ or $\mathbb{C}$ when counting the roots with their multiplicity. But does it split over $\mathbb{R}$ or $\mathbb{C}$?
For example the polynomial $x^2 + 1$ does not have any real roots and so it does not split across $\mathbb{R}$. It does however have two different complex roots so it splits across $\mathbb{C}$. The following theorem, that we will accept as true, allows us to say that all complex coefficient polynomials split across $\mathbb{C}$.

Theorem 1.14 (de d'Alembert's Theorem). *Any polynomial $P \in \mathbb{C}[x]$ such that $d(P) \geq 1$ has at least one root.*

Corollary 4. *Let $P \in \mathbb{C}[x]$ a polynomial with degree $n \geq 1$. Then P has exactly n roots. Some may be distinct, some may be equal. In other words, P splits across $\mathbb{C}$.*

Proof We prove the theorem by induction on the degree $n \geq 1$ of $P \in \mathbb{C}[x]$.
Induction statement : Let $n \in \mathbb{N}^*$ and $\mathcal{P}(n)$ the proposition: "Any polynomial $P \in \mathbb{C}[x]$ of degree n has exactly n roots counted with their multiplicity. "

Base case : $n = 1$: P has at most one root as its degree is 1 and P has at least one root according to the previous theorem. So $\mathcal{P}(1)$ holds.

Inductive step : We suppose that there exists $n \in \mathbb{N}^*$ such that $\mathcal{P}(n)$ is true and we show that $\mathcal{P}(n+1)$ is true.

To do so we consider $P \in \mathbb{C}[x]$ of degree $n+1$. According to the previous theorem, P has at least one root $\alpha \in \mathbb{C}$. So $P(x) = (x-\alpha)Q(x)$ with Q of degree n.
We have that

$$a \text{ is a root of } P \Leftrightarrow a = \alpha \text{ or } a \text{ root of } Q.$$

However according to the induction hypothesis, Q has exactly n roots counted with their multiplicity. As a consequence, P has exactly $n+1$ roots. The induction holds and the theorem is thus proven. $\square$

1.1.5 Complex roots of real polynomials

We consider a n^{th} degree ($n \geq 1$) polynomial P of $\mathbb{R}[x]$. As P can also be viewed as a polynomial of $\mathbb{C}[x]$, it has n different or equal roots in $\mathbb{C}$ (according to corollary 4).

Theorem 1.15. *Let P be a polynomial of $\mathbb{R}[x]$ of degree greater or equal to 2 and α a complex root of P of multiplicity $k \geq 1$ with $Im(\alpha) \neq 0$. Then, $\bar{\alpha}$ is also a root of P of multiplicity k.*

Proof We denote $P(x) = a_n\, x^n + \ldots + a_0$. By hypothesis,

$$P(\alpha) = a_n\, \alpha^n + \ldots + a_0 = 0.$$

As the conjugate of the sum of complex numbers is equal to the sum of the conjugates and the conjugate of the product is equal to the product of the conjugates, we also have that

$$P(\overline{\alpha}) = a_n\, \overline{\alpha}^n + \ldots + a_0 = \overline{\overline{a_n}\alpha^n + \ldots + \overline{a_0}} = \overline{a_n\alpha^n + \ldots + a_0},$$

because the coefficients $a_0, \ldots, a_n$ are real numbers. We deduce that

$$P(\overline{\alpha}) = \overline{P(\alpha)} = \overline{0} = 0.$$

To conclude, $\overline{\alpha}$ is a root of P. We can then write,

$$P(x) = (x-\alpha)\,(x-\overline{\alpha})\,Q(x) = (x^2 - 2\ \mathrm{Re}(\alpha)x + |\alpha|^2)\,Q(x), \tag{1.2}$$

where $Q \in \mathbb{R}[x]$ is the quotient of the Euclidean division of P by $x^2 - 2\ \mathrm{Re}(\alpha)x + |\alpha|^2 \in \mathbb{R}[x]$.
If α is a multiple root of P, we repeat the process with $Q(x)$ as α is a root of Q. $\square$

Corollary 5. *If the degree of $P \in \mathbb{R}[x]$ is odd and worth $n = 2m+1$, the polynomial has at least one real root.*

Proof According to the previous property, complex roots are pairs of conjugates. As a consequence any polynomial P with real coefficients has an even number of non real roots. Thus, a polynomial P belonging to $\mathbb{R}[x]$ with no real roots has an even degree. We obtain the result as the converse statement. $\square$

1.1.6 Polynomial factorisation into the product of irreducible polynomials

Definition 1.16. *An* **irreducible polynomial** *is a polynomial A of degree ≥ 1 such that if a polynomial B divides A then there exists $\alpha \in \mathbb{K}$ such that $B = \alpha$ or $B = \alpha A$.*
In other words a polynomial of degree ≥ 1 is irreducible if it cannot be factorised into two polynomials of positive degree.

Remark 4. *To know whether a polynomial is irreducible or not, one has to try to factorise it. One method is thus to search for its real or complex roots. So the form of an irreducible polynomial differs when working in $\mathbb{R}[x]$ or in $\mathbb{C}[x]$.*

According to de d'Alembert's theorem, **monic** and **irreducible** polynomials over $\mathbb{C}[x]$ are polynomials of the form $x-\alpha$ with $\alpha \in \mathbb{C}$. The following theorem gives the form of irreducible polynomials over $\mathbb{R}[x]$

Theorem 1.17. *The irreducible polynomials over $\mathbb{R}[x]$ are*

- *the polynomials of degree 1,*
- *the polynomials of degree 2 with no real roots.*

Proof It is clear that the two given types of polynomials are irreducible over $\mathbb{R}[x]$ and that they are the only irreducible polynomials of degree lesser or equal to 2. It remains to show that polynomials of degree greater to 2 cannot be irreducible.
Let P be a polynomial of degree $n > 2$

- if P has a real root (denoted α) then there exists $Q \in \mathbb{R}[x]$ of degree $n-1 > 0$ such that $P(x) = (x-\alpha)Q(x)$. P is thus not irreducible.
- if P has no real roots then according to de d'Alembert's theorem, it has a complex root α. Furthermore according to Theorem 1.15, $\bar{\alpha}$ is also a root of P. Thus there exists $Q \in \mathbb{R}[x]$ of degree $n-2 > 0$ such that

  $$P(x) = (x-\alpha)(x-\bar{\alpha})Q(x) = (x^2 - 2Re(\alpha)x + |\alpha|^2)Q(x)$$

  and P is not irreducible.

$\square$

Theorem 1.18. *1. If $P \in \mathbb{C}[x]$ is of degree n, then it can be written in a single way (up to the order of the factors) under the form:*

$$\boxed{P(x) = a_n\,(x-\alpha_1)^{k_1}\,\ldots\,(x-\alpha_p)^{k_p}} \tag{1.3}$$

where $\alpha_1,\,\ldots,\,\alpha_p$ are complex numbers and $k_1,\ldots,k_p$ are positive integrers. Moreover, we have that $k_1+\ldots+k_p=n$.

2. If $P \in \mathbb{R}[x]$ is of degree n, then it can be written in a single way (up to the order of the factors) under the form:

$$\boxed{P(x) = a_n\,(x-\alpha_1)^{k_1}\,\ldots(x-\alpha_l)^{k_l}\,(x^2-s_1\,x+p_1)^{t_1}\,\ldots(x^2-s_m\,x+p_m)^{t_m},} \tag{1.4}$$

with

(i) α_i, s_i et p_i real numbers such that $s_i^2-4\,p_i<0$,

(ii) k_i, t_i positive integers. Furthermore, we have that $k_1+\ldots+k_l+2\,(t_1+\ldots+t_m)=n$.

We obtain the form (1.4) by grouping together two by two the factors of the form $(x-\alpha)\,(x-\overline{\alpha})$ in factorisation (1.3).
Form 1.3 (respectively (1.4)) is called the **factorisation into a product of irreducible polynomials** of polynomial P over $\mathbb{C}[x]$ (respectively over $\mathbb{R}[x]$).

Example 7.

1. *The factorisation into a product of irreducible polynomials of $P(x)=x^4-1$ is*
   - $P(x)=(x-1)\;(x+1)\;(x^2+1)$ *over* $\mathbb{R}[x]$,
   - $P(x)=(x-1)\;(x+1)\;(x-i)\;(x+i)$ *over* $\mathbb{C}[x]$.
2. *The factorisation into a product of irreducible polynomials of $P(x)=x^4+2\,x^2+1$ is*
   - $P(x)=(x^2+1)^2$ *over* $\mathbb{R}[x]$,
   - $P(x)=(x-i)^2\;(x+i)^2$ *over* $\mathbb{C}[x]$.
3. *The factorisation into a product of irreducible polynomials of $P(x)=x^4-2\,x^2+1$ is $P(x)=(x+1)^2\;(x-1)^2$ over both $\mathbb{R}[x]$ and $\mathbb{C}[x]$.*

Example 8. *Factor into a product of irreducible polynomials over both $\mathbb{R}[x]$ and $\mathbb{C}[x]$ the following polynomials: x^4-4, x^4+1, x^4+4x^2+4 et x^4-4x^2+4.*

□

1.2 Rational functions

1.2.1 Definition

In general, the quotient of two polynomial functions is not a polynomial function. This leads us to introduce a new family of functions: rational functions. The latter are frequently used in some engineering fields like for example signal processing. From a mathematical point of view, we mainly need to decompose them into *partial fractions* in order to integrate them. (cf paragraph 1.2.2).

Definition 1.19. *A* **rational function** *or* **rational fraction** *is a function of the form* $F = \dfrac{P}{Q}$ *where* P *and* Q *are polynomial functions with* $Q \neq 0_{\mathbb{K}(x)}$*. The rational function* F *is null if* $P = 0_{\mathbb{K}[x]}$*. We denote* $\mathbb{K}(x)$ *the set of rational functions over* $\mathbb{K}$ *(*$\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*) :*

$$\mathbb{K}(x) = \left\{ \frac{P}{Q},\ (P,\, Q) \in (\mathbb{K}[x])^2,\ \ Q \neq 0_{\mathbb{K}[x]} \right\}. \tag{1.5}$$

Before starting to study some properties of these functions, we must clear up their definition domain. By convention, the definition domain of a rational fraction will be that of its irreducible form (see definition 1.20 and the following example).

Example 9.

1. *Let* $P(x) = x^2 + x - 2$ *and* $Q(x) = x^2 + 5x + 6$*. The function* $F(x) = \dfrac{P(x)}{Q(x)}$ *of* $\mathbb{R}[x]$ *can be written under the form*

   $$F(x) = \frac{x^2 + x - 2}{x^2 + 5x + 6}. \tag{1.6}$$

   F *is not defined on the roots of* Q*, i.e.* -2 *and* -3*, because* $Q(x) = (x+2)\,(x+3)$*. But* $P(x) = (x-1)\,(x+2)$ *so* -2 *is also a root of* P*. For any value of* x *different to 2, we can simplify* F *be dividing both its numerator and denominator by* $x + 2$ *and*

   $$F(x) = \frac{x-1}{x+3} = \frac{P_1(x)}{Q_1(x)}. \tag{1.7}$$

   In expression (1.6)*,* P *and* Q *share a common root whereas in* (1.7)*,* P_1 *and* Q_1 *have no common roots.*

2. *Simplify* $G(x) = \dfrac{x^2 + x - 2}{x^2 - 2x + 1}$.

Definition 1.20. *The* **irreducible form** *of a rational function* $F \in \mathbb{K}(x)$ *is any expression* $F = \dfrac{P}{Q}$ *where* P *and* Q *are two polynomial functions with no shared root in* $\mathbb{C}$*. This form is unique for any non zero rational function if* Q *is a monic polynomial.*

Definition 1.21. *Let* $F = \dfrac{P}{Q}$ *a rational function over* $\mathbb{K}$ *written under an irreducible form*

- *the roots of* P *are the* **zeros** *of* F*,*

- *the roots of* Q *are the* **poles** *of* F*.*

If F is a rational function with real coefficients, its **definition domain** is the set $\mathbb{R}$ without its real poles.

Definition 1.22. *Let* $F = \dfrac{P}{Q}$ *a rational function written under an irreducible form.*
When performing Euclidean division of P *by* Q*, we obtain two polynomials* E *and* R *with* $d(R) < d(Q)$ *such that* $P = EQ + R$*. So the rational function can be written as*

$$\frac{P}{Q} = \frac{EQ+R}{Q} = E + \frac{R}{Q}.$$

The polynomial E *is called the integer part of* F*. Through uniqueness of the quotient and the remainder of Euclidean division, the integer part of a rational function* $F = \dfrac{P}{Q}$ *is zero if* $d(P) < d(Q)$*.*

In other words, a rational function can always be written as the sum of a polynomial, sometimes null, that is its integer part, and an irreducible fraction R/Q such that the degree of the numerator is lesser than the degree of the denominator.

Example 10. *Let* $F(x) = \dfrac{x^2 - 3\,x + 1}{x - 2}$*. As the degree of the numerator is greater than that of the denominator,* F *has a non zero integer part. It can be computed through the Euclidean division of* $x^2 - 3\,x + 1$ *by* $x - 2$*. We have that*

$$x^2 - 3\,x + 1 = (x-2)\,(x-1) - 1.$$

The integer part of F *is therefore* $x - 1$*, and* $\dfrac{x^2 - 3\,x + 1}{x - 2} = x - 1 - \dfrac{1}{x-2}$*.*

1.2.2 Partial fraction decomposition

Partial fraction

Definition 1.23. *A* **partial fraction** *is a rational function of the form*

$$F = \frac{A}{Q^k}, \qquad k \geq 1,$$

with Q *an irreducible monic polynomial and* $d(A) < d(Q)$*.*

The partial fractions depend on the form of the polynomial Q.

In $\mathbb{C}[x]$, there is a **single type** of irreducible polynomial and thus a single type of partial fraction:

- Irreducible monic polynomial: $x - \alpha$, $(\alpha \in \mathbb{C})$,

- Partial fractions:

$$\boxed{\frac{a}{(x-\alpha)^k}, \qquad a \text{ and } \alpha \text{ in } \mathbb{C}}$$

In $\mathbb{R}[x]$,there are **two types** of irreducible polynomials and thus two types of partial fractions:

1. Partial fraction for an irreducible first order polynomial with real coefficients:

    - Irreducible monic polynomial: $x - \alpha$, $(\alpha \in \mathbb{R})$,
    - Partial fractions:

$$\boxed{\frac{a}{(x-\alpha)^k}, \qquad a \text{ and } \alpha \text{ in } \mathbb{R}}$$

2. Partial fraction for an second order polynomial with real coefficients, irreducible over $\mathbb{R}$:

    - Irreducible monic polynomial: $x^2 + px + q$, $(p,\ q \in \mathbb{R}$ tels que $p^2 - 4q < 0)$,
    - Partial fractions:

$$\boxed{\frac{ax+b}{(x^2+px+q)^k}, \qquad a,\ b,\ p,\ q \in \mathbb{R} \text{ such that } p^2 - 4q < 0}$$

General principle

Let $F = \dfrac{P}{Q}$ be a rational function written under its irreducible form. We note that Q is monic.

1. We first perform Euclidean division of P by Q. We then obtain $F = E + \frac{P_1}{Q}$ with $d(P_1) < d(Q)$. We note that this step is of interest only when $d(P) \geq d(Q)$ because if $d(P) < d(Q)$, $E = 0$ and $P_1 = P$.

2. We factorise Q into a product of monic and irreducible factors over $\mathbb{K}$. The factorisation is different when working over $\mathbb{C}$ or over $\mathbb{R}$, so we distinguish two different cases

    (a) *Decomposition over* $\mathbb{C}$ *:* In this case,

$$Q = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2}\dots(x-\alpha_\ell)^{m_\ell}.$$

The decomposition of F into partial fractions over $\mathbb{C}$ writes as:

$$\begin{aligned} F &= E + \frac{a_{1,1}}{x-\alpha_1} + \ldots + \frac{a_{1,m_1}}{(x-\alpha_1)^{m_1}} \\ &\quad + \quad \ldots \quad \ldots \\ &\quad + \frac{a_{\ell,1}}{x-\alpha_\ell} + \ldots + \frac{a_{\ell,m_\ell}}{(x-\alpha_\ell)^{m_\ell}} \\ &= E + \sum_{k=1}^{\ell} \left(\frac{a_{k,1}}{x-\alpha_k} + \ldots + \frac{a_{k,m_k}}{(x-\alpha_k)^{m_k}} \right). \end{aligned}$$

where the coefficients $a_{k,j}$ are complex numbers. We can check that this expression is **unique**.

(b) *Decomposition over* $\mathbb{R}$ *:* In this case,

$$Q = (x-\alpha_1)^{m_1}(x-\alpha_2)^{m_2} \ldots (x-\alpha_\ell)^{m_\ell}(x^2 - s_1 x + p_1)^{t_1} \ldots (x^2 - s_r x + p_r)^{t_r}.$$

The decomposition of F into partial fractions over $\mathbb{R}$ becomes:

$$\begin{aligned} F = E &+ \sum_{k=1}^{\ell} \left(\frac{a_{k,1}}{x-\alpha_k} + \ldots + \frac{a_{k,m_k}}{(x-\alpha_k)^{m_k}} \right) \\ &+ \sum_{k=1}^{r} \left(\frac{b_{k,1}x + c_{k,1}}{x^2 - s_k x + p_k} + \ldots + \frac{b_{k,t_k}x + c_{k,t_k}}{(x^2 - s_k x + p_k)^{t_k}} \right). \end{aligned}$$

The coefficients $a_{k,j}$, $b_{k,j}$ and $c_{k,j}$ are real numbers here and we have uniqueness of the decomposition once again.

In practice

In this part we go through the steps and main techniques that allows us to compute a partial fraction decomposition.

It is important to bare in mind that several methods exist to compute the decomposition's coefficients and that the choice and order of use of these methods is not always the same. One should try to choose the method that requires the least computations depending on the form of the rational function.

1. Writing the general form of the decomposition.
   This first step requires:

   (a) to compute the integer part of P/Q,

   (b) to factorise Q into a product of irreducible factors over $\mathbb{R}[x]$ or $\mathbb{C}[x]$ depending on whether we wish to decompose over $\mathbb{R}[x]$ or $\mathbb{C}[x]$.

   One should be wary of some traps like for example :

$$F_1(x) = \frac{1}{(x+1)(x^2-1)} = \frac{1}{(x+1)^2(x-1)}.$$

The decomposition of F_1 thus has the following form:

$$F_1(x) = \frac{a_1}{(x+1)^2} + \frac{b_1}{(x+1)} + \frac{c_1}{(x-1)}.$$

Below we give some more examples:

$$F_2(x) = \frac{1}{x^2\,(x^2+1)} = \frac{a_2}{x} + \frac{b_2}{x^2} + \frac{c_2 x + d_2}{x^2+1} \text{ over } \mathbb{R}[x],$$

$$F_2(x) = \frac{1}{x^2\,(x^2+1)} = \frac{a_2}{x} + \frac{b_2}{x^2} + \frac{e_2}{x-i} + \frac{f_2}{x+i} \text{ over } \mathbb{C}[x],$$

$$F_3(x) = \frac{1}{x^3\,(x^2+1)} = \frac{a_3}{x} + \frac{b_3}{x^2} + \frac{c_3}{x^3} + \frac{d_3\,x + e_3}{x^2+1} \text{ over } \mathbb{R}[x].$$

$$F_4(x) = \frac{x+2}{(x-1)\,(x+3)^3} = \frac{a_4}{x-1} + \frac{b_4}{x+3} + \frac{c_4}{(x+3)^2} + \frac{d_4}{(x+3)^3}.$$

$$F_5(x) = \frac{x^2+1}{x\,(x^2+x+1)^2} = \frac{a_5}{x} + \frac{b_5\,x + c_5}{x^2+x+1} + \frac{d_5\,x + e_5}{(x^2+x+1)^2}.$$

2. Coefficient computation:
   The first common trap is to try to reduce to a common denominator. Indeed this method generally leads to long and complicated computations. Below we give some more efficient methods:

   (a) Fraction parity:
   When F is odd or even, we can find relationships between the coefficients that simplify the calculations.

   i. F_2 is even, so for any $x \in \mathbb{R}^*$, $F_2(x) = F_2(-x)$; when searching for its decomposition over $\mathbb{R}[x]$ we will be able to write:

   $$\frac{a_2}{x} + \frac{b_2}{x^2} + \frac{c_2 x + d_2}{x^2+1} = \frac{-a_2}{x} + \frac{b_2}{x^2} + \frac{-c_2 x + d_2}{x^2+1}.$$

   Through uniqueness of the decomposition we deduce that $a_2 = -a_2$ and $c_2 = -c_2$, so that $a_2 = c_2 = 0$. It remains to find b_2 and d_2 such that

   $$F_2(x) = \frac{b_2}{x^2} + \frac{d_2}{x^2+1}.$$

   Note that we would have reached the same conclusion by using that $F_2(x) = G(x^2)$ with $G(u) = \dfrac{1}{u\,(u+1)}$.

   ii. F_3 is odd and verifies $F_3(x) = -F_3(-x)$ for any non zero x. We deduce that

   $$F_3(x) = \frac{a_3}{x} + \frac{c_3}{x^3} + \frac{d_3\,x}{x^2+1}.$$

(b) Multiplying by $(x-\alpha)^k$:
For a pole α of multiplicity k, we start by computing the coefficient of the term $\dfrac{1}{(x-\alpha)^k}$. To do so, we multiply the function by $(x-\alpha)^k$ then we simplify the expression before computing its value for $x=\alpha$:

i. When computing $G_1(x)=(x+1)^2\,F_1(x)$, we obtain the equality

$$G_1(x)=\frac{1}{(x-1)}=a_1+b_1\,(x+1)+\frac{c_1\,(x+1)^2}{(x-1)},$$

and then $a_1=\dfrac{1}{-1-1}=-\dfrac{1}{2}$ is obtained by calculating $G_1(-1)$.

ii. When computing $(x-1)\;F_1(x)$, we obtain the equality

$$\frac{1}{(x+1)^2}=\frac{a_1\,(x-1)}{(x+1)^2}+\frac{b_1\,(x-1)}{(x+1)}+c_1,$$

then, taking the value at $x=1$, $c_1=\dfrac{1}{(1+1)^2}=\dfrac{1}{4}$.
Through these two steps we have simplified the problem. It remains to compute b_1 such that

$$F_1(x)=-\frac{1}{2\,(x+1)^2}+\frac{b_1}{(x+1)}+\frac{1}{4\,(x-1)}$$

iii. When applying this method to the other functions, we obtain that
$F_2(x)=\dfrac{1}{x^2}+\dfrac{d_2}{x^2+1}$ over $\mathbb{R}[x]$,
$F_2(x)=\dfrac{1}{x^2\;(x^2+1)}=\dfrac{1}{x^2}-\dfrac{1}{2i\,(x-i)}+\dfrac{1}{2i\,(x+i)}$ over $\mathbb{C}[x]$, and our job is done. We can use this result to find the decomposition of F_2 over $\mathbb{R}[x]$,

$$\boxed{\frac{1}{x^2\;(x^2+1)}=\frac{1}{x^2}-\frac{1}{x^2+1}. \qquad (1.8)}$$

$F_3(x)=\dfrac{1}{x^3\,(x^2+1)}=\dfrac{a_3}{x}+\dfrac{1}{x^3}+\dfrac{d_3\,x}{x^2+1}$ over $\mathbb{R}[x]$.
$F_4(x)=\dfrac{3}{64\;(x-1)}+\dfrac{b_4}{x+3}+\dfrac{c_4}{(x+3)^2}+\dfrac{1}{4\,(x+3)^3}$.
$F_5(x)=\dfrac{1}{x}++\dfrac{b_5\,x+c_5}{x^2+x+1}+\dfrac{d_5\,x+e_5}{(x^2+x+1)^2}$.

(c) Limit value at $+\infty$:
We can obtain relations between remaining unknown coefficients by multiplying $F(x)$ by x or x^2 before taking x to $+\infty$.

$\lim\limits_{x\to\infty} x\,F_1(x)=0=b_1+\dfrac{1}{4}$ gives the value of b_1 and allows us to conclude,

$$\boxed{\frac{1}{(x+1)^2(x-1)} = -\frac{1}{2\,(x+1)^2} - \frac{1}{4\,(x+1)} + \frac{1}{4\,(x-1)}.}$$

$\lim\limits_{x\to\infty} x^2\,F_2(x) = 0 = 1 + d_2$ gives $d_2 = -1$, and confirms the expression given in (1.8).

$\lim\limits_{x\to\infty} x\,F_3(x) = 0 = a_3 + d_3$, so $d_3 = -a_3$.

$\lim\limits_{x\to\infty} x\,F_4(x) = 0 = \dfrac{3}{64} + b_4$, then gives

$$F_4(x) = \frac{3}{64\ (x-1)} - \frac{3}{64\ (x+3)} + \frac{c_4}{(x+3)^2} + \frac{1}{4\,(x+3)^3}.$$

$\lim\limits_{x\to\infty} x\,F_5(x) = 0 = 1 + b_5$ gives $b = -1$.

(d) Specific values :
We obtain other relations between coefficients by taking specific values for x (different from the poles!)

$F_3(1) = \dfrac{1}{2} = a_3 + 1 + \dfrac{d_3}{2}$, with $d_3 = -a_3$ allows us to conclude:

$$\boxed{\frac{1}{x^3\,(x^2+1)} = -\frac{1}{x} + \frac{1}{x^3} + \frac{x}{x^2+1}.}$$

$F_4(-2) = 0 = -\dfrac{1}{64} - \dfrac{3}{64} + c_4 + \dfrac{1}{4}$ also allows us to conclude:

$$\boxed{\frac{x+2}{(x-1)\ (x+3)^3} = \frac{3}{64\ (x-1)} - \frac{3}{64\ (x+3)} - \frac{3}{16\,(x+3)^2} + \frac{1}{4\,(x+3)^3}}$$

$F_5(x) = \dfrac{1}{x} + \dfrac{-x+c_5}{x^2+x+1} + \dfrac{d_5\,x+e_5}{(x^2+x+1)^2}$ still has 3 unknown coefficients. We have to choose 3 different and specific values to obtain a linear system for which the unknowns are the 3 coefficients. By choosing for example the values -1, 1 and -2, we are led to the system:

$$\left\{\begin{array}{llllll} c_5 & -d_5 & +e_5 & = & -2 \\ 3\,c_5 & +d_5 & +e_5 & = & -4 \\ 3\,c_5 & -2\,d_5 & +e_5 & = & -4 \end{array}\right.$$

By subtracting the third equation from the second, we obtain $d_5 = 0$. We then insert this value into the first and second equations:

$$\left\{\begin{array}{llll} c_5 & +e_5 & = & -2 \\ 3\,c_5 & +e_5 & = & -4 \end{array}\right.$$

Subtraction gives $c_5 = -1$, then $e_5 = -1$. We conclude that:

$$\boxed{\frac{x^2+1}{x\ (x^2+x+1)^2} = \frac{1}{x} - \frac{x+1}{x^2+x+1} - \frac{1}{(x^2+x+1)^2}.}$$

Remark 5. *As mentioned previously, partial fraction decomposition is a tool that allows us to compute integrals. For instance suppose that we wish to compute*

$$\int_0^1 \frac{4x^2 - 4x + 2}{(x-1)(x^2+1)} dx.$$

It would not be easy to find a primitive for the function $f(x) = \frac{4x^2-4x+2}{(x-1)(x^2+1)}$*. However partial fraction composition gives*

$$\frac{4x^2 - 4x + 2}{(x-1)(x^2+1)} = \frac{2}{x-2} + \frac{2x}{x^2+1}.$$

Therefore

$$\int_0^1 \frac{4x^2 - 4x + 2}{(x-1)(x^2+1)} dx = \int_0^1 \frac{2}{x-2} + \frac{2x}{x^2+1} dx.$$

Given this latter form, a primitive of f *is easy to compute and thus:*

$$\int_0^1 \frac{4x^2 - 4x + 2}{(x-1)(x^2+1)} dx = \left[2\ln|x-2| + \ln|x^2+1|\right]_0^1 = -\ln(2).$$

We will see more detail on how to integrate partial fractions in the chapter on integration.

1.3 Recap quiz

Polynomials:

1. Let $P(x) = -2x + 7x^3 - x^5$ and $Q(x) = 1 - x^2 + x^3$. Give the degree and valuation of P. Is P a monic polynomial? Same questions for Q.
2. Give the degree and valuation of a constant non zero polynomial. Give the degree and valuation of the zero polynomial.
3. Recall a criterion that gives equality between two polynomials.
4. Let P_1 and P_2 in $\mathbb{K}[x]$. Write the Euclidean division of polynomial P_1 by polynomial P_2.
5. Let $P \in \mathbb{K}[x]$ and $\alpha \in \mathbb{K}$. Give two equivalent characterisations of the statement "α is a root of P".
6. Let $P \in \mathbb{K}[x], \alpha \in \mathbb{K}$ and $k \in \mathbb{N}$. What does "α is a root of multiplicity k of P" mean?
7. What is the link between the degree of a polynomial P and its number of roots (different or not)?
8. Let $P \in \mathbb{K}[x]$. What does "P splits over $\mathbb{K}$" mean?
9. Let $(\alpha, \beta) \in \mathbb{R}^2$. Give a monic second degree polynomial with real coefficients for which the roots are α and β.

10. Let $\alpha \in \mathbb{C} \setminus \mathbb{R}$. Give a monic second degree polynomial with real coefficients for which α is root. Give the other root of the polynomial.

11. Give the complex roots of $P(x) = (x^2+1)(x^2+x+1)(x^2-x+1)$.

12. Let $P \in \mathbb{R}[x]$ such that $\deg P = 4$ for which $\alpha = 1+2i$ is double root. Give the other roots of P and find the coefficients of P.

13. Let $\mathcal{F} = \{P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8\}$ a family of polynomials defined by

$$P_1(x) = x - 7, \quad P_2(x) = x^2 - 25, \quad P_3(x) = x^3 - 8, \quad P_4(x) = x^4 + 16,$$

$$P_5(x) = x + 7, \quad P_6(x) = x^2 + 25, \quad P_7(x) = x^3 + 8, \quad P_8(x) = x^4 - 16.$$

    (a) List the polynomials of $\mathcal{F}$ that are irreducible over $\mathbb{R}$. Factorise the remaining polynomials into products of irreducible polynomials over $\mathbb{R}$.

    (b) List the polynomials of $\mathcal{F}$ that are irreducible over $\mathbb{C}$. Factorise the remaining polynomials into products of irreducible polynomials over $\mathbb{C}$.

Rational functions:

1. What is a rational function?

2. What is an irreducible rational function?

3. When is the integer part of an irreducible rational function zero?

4. When is the integer part of an irreducible rational function not zero? In this case, give the degree of the integer part in function of that of the numerator and the denominator.

5. Give the general form of partial fractions over $\mathbb{R}[x]$.

6. Give the general form of partial fractions over $\mathbb{C}[x]$.

7. Let F be an irreducible rational function with coefficients over K. Define what a zero of F is. Define what a pole of F is.

8. Let $G_1(x) = \dfrac{x^2(x-1)}{x^4 - 2x^2 + 1}$. Give the real-valued zeros and poles of G_1 and specify their multiplicity.

9. Let $G_2(x) = \dfrac{(x-1)}{x(x^2+1)}$. Give the real-valued zeros and poles of G_1 and specify their multiplicity.

10. Decompose the following into partial fractions over $\mathbb{R}$: $G_3(x) = \dfrac{2x}{x^2-25}$ and $G_4(x) = \dfrac{12}{x^3-8}$.

11. Let $G_5(x) = \dfrac{x^5}{(x^2+25)^2}$. Give, without computing the coefficients, the form of the decomposition of G_5 into partial fractions over $\mathbb{R}$ and $\mathbb{C}$.

12. Let $G_6(x) = \dfrac{x+2}{(x^3+8)^2}$. Give the real-valued poles of G_4 and their multiplicity. Give, without computing the coefficients, the form of the decomposition of G_6 into partial fractions over $\mathbb{R}$ and $\mathbb{C}$.

Chapter 2

Limits, continuity and differentiability

2.1 Intervals

2.1.1 Definitions

Definition 2.1. *For two real numbers a and b such that $a \leq b$, the* segment $[a, b]$ *is the set of real numbers valued between a at b*

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\} .$$

A set in $\mathbb{R}$, denoted I, is an interval *if*

$$(\forall (a, b) \in I^2)(a < b \Longrightarrow [a, b] \subset I).$$

As the empty set $\emptyset$ does not contain any elements, it can be considered that it verifies the previous property and is thus an interval.

The proof of the following proposition, giving the main properties of intervals, is a direct consequence of Definition 2.1.

Proposition 2.2 (Intervals of $\mathbb{R}$)**.** *Let $a \leq b$ two real numbers. Each of the following sets of $\mathbb{R}$ are intervals:*

- ***minimal interval*** *: $\emptyset$ and* ***maximal interval*** *(or* **real line***) : $\mathbb{R}$;*
- ***closed unbounded intervals*** *(or* ***closed rays*** *) $(-\infty, a]$ and $[a, +\infty)$;*
- ***open unbounded intervals*** *(or* ***open rays****) $(-\infty, a)$ and $(a, +\infty)$;*
- ***closed bounded intervals*** *(or* ***segments****) $[a, b]$;*
- ***open bounded intervals****: $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$;*
- ***semi-open*** *(or* ***semi-closed****) bounded intervals: $(a, b]$, $[a, b)$.*

Exercise 1. *Among the following sets, which ones are intervals of $\mathbb{R}$?*

$$\begin{array}{ll} I_1 = [1, 2) , & I_2 = (-1, 1) , \\ I_3 = (-\infty, 0) , & I_4 = [3, +\infty) , \\ I_5 = (0, 1) \cup (1, 3) . & \end{array}$$

Remark 1. *Any segment* $[a, b]$ *can be written using the* ***unit segment*** $[0, 1]$ *as follows*

$$[a, b] = \{x \in \mathbb{R} \mid \exists t \in [0, 1], \quad x = (1-t)a + tb\} \ ;$$

furthermore, if $a < b$*, the function that maps* t *to* $(1-t)a+tb$ *is a bijection of* $[0, 1]$ *to* $[a, b]$.

Proposition 2.3. *Let* I *and* J *be two real intervals. Then*

- $I \cap J$ *is an interval ;*
- ($I \cap J \neq \emptyset \Rightarrow I \cup J$ *is an interval*).

Proof

- We show that for any pair (x, y) of reals from $I \cap J$ with $x < y$, the set $[x, y]$ belongs to $I \cap J$. Let $(x, y) \in (I \cap J)^2$. We have that $(x, y) \in I^2$ and that I is an interval so $[x, y] \subset I$. Moreover $(x, y) \in J^2$ and J is an interval so $[x, y] \subset J$. It follows that $[x, y] \subset I \cap J$ so $I \cap J$ is an interval.

- $\forall (x, y) \in (I \cup J)^2$ with $x < y$, ($x \in I$ or $x \in J$) and ($y \in I$ or $y \in J$). Through symmetry, we are able to solely consider the following two cases:

  - $x \in I$ and $y \in I$. In this case, as I is an interval, $[x, y] \subset I \subset I \cup J$,
  - $x \in I \setminus J$, $y \in J \setminus I$. In this case, as $I \cap J \neq \emptyset$, we claim that there exists $z \in I \cap J$ such that $x < z < y$. Indeed, if for some $z \in I \cap J, z \leq x < y$ then as J is an interval and $(z, y) \in J^2$ we deduce that $x \in J$ which is absurd. By the same reasoning, it is absurd to suppose that $x < y \leq z$. So there exists $z \in I \cap J$ such that $x < z < y$ and $(x, z) \in I^2$ and $(y, z) \in J^2$ so as I and J are intervals, $[x, z] \subset I \subset I \cup J$ and $[z, y] \subset J \subset I \cup J$. As $[x, z] \cup [z, y] = [x, y]$ we then have $[x, y] \subset I \cup J$.

$\square$

Beware: The converse is not true. $]0, 1] \cup]1, 2]$ is an interval but $]0, 1] \cap]1, 2] = \emptyset$.

2.1.2 Mathematical neighbourhood

The idea of a mathematical neighbourhood is based on the following open intervals defined for $a \in \mathbb{R}$ and $\ell > 0$

$$(a - \ell, a + \ell) = \{x \in \mathbb{R} \ , \quad |x - a| < \ell\} \, .$$

This interval is called the **open interval** whose **center** is a and whose **radius** is ℓ.

Definition 2.4. $V \subset \mathbb{R}$ *is said to be a* ***neighbourhood*** *of the point* $a \in \mathbb{R}$ *if there exists* $\ell > 0$ *such that* $]a - \ell, a + \ell[\subset V$*, in other words if* V *contains an open interval whose center is* a. ***The set of neighbourhoods*** *of* a *is denoted* $\mathcal{V}(a)$.

It can sometimes be useful, to describe limiting behaviours for example, to give a name to $\mathbb{R}$ to which we add $+\infty$ and $-\infty$. The resulting object is called the **extended real number line**, denoted $\overline{\mathbb{R}}$. A number $x \in \mathbb{R}$ will then be said to be **finite** as opposed to the two symbols $+\infty$ and $-\infty$. Thus, writing $a \in \overline{\mathbb{R}}$ means that a is either a finite number, or $+\infty$ or $-\infty$.
It is important to note however that, in general, standard arithmetic operations do not extend to $+\infty$ and $-\infty$. We will come back to this later.

Definition 2.5. *It is said that $V \subset \mathbb{R}$ is a neighbourhood of $+\infty$ (or that $V \in \mathcal{V}(+\infty)$) if there exists $A \in \mathbb{R}$ such that $(A, +\infty) \subset V$.*

Similarly, $V \subset \mathbb{R}$ is a neighbourhood of $-\infty$ (or $V \in \mathcal{V}(-\infty)$) if there exists $A \in \mathbb{R}$ such that $(-\infty, A) \subset V$.

One can easily check that

$$\forall a \in \overline{\mathbb{R}},\ \forall (V, V') \in (\mathcal{V}(a))^2 \text{ then } V \cup V' \in \mathcal{V}(a) \text{ and } V \cap V' \in \mathcal{V}(a).$$

2.2 Limit of a function

2.2.1 Definitions and basic properties

In the following, when we say that a function is defined on a neighbourhood of a, we imply that the function may possibly not be defined at point a itself.

Finite limit when the variable tends to a finite value

We start off by discussing several cases and giving a definition of limit in each of these cases. We will see at the end of this paragraph how the notion of neighbourhood allows us to unify all these definitions.

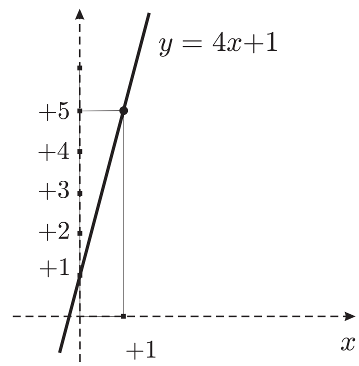
Definition 2.6. *We say that the function f **tends to** ℓ when x tends to a (or **the limit of f** at $x = a$ is equal to ℓ) when the function is defined on a neighbourhood of a (except possibly at point a itself) and when it verifies*

$$\boxed{\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall x \in \mathcal{D}_f, \quad 0 < |x - a| \leq \eta \implies |f(x) - \ell| \leq \varepsilon\,.} \tag{2.1}$$

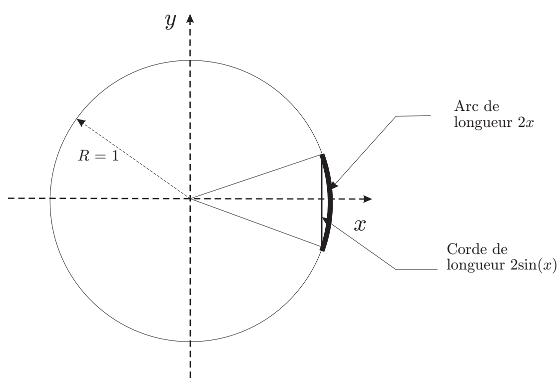
In other words, f tends to ℓ when x tends to a when f is defined on a neighbourhood of a and we can make $|f(x) - \ell|$ as small as we want as long as $|x - a|$ is small enough.

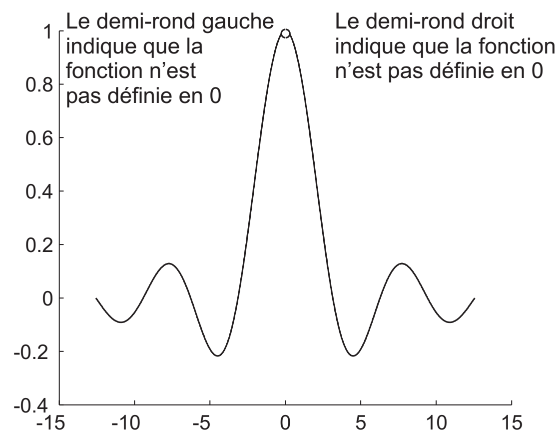
Example 2. *The following examples show different situations that we can encounter.*

1. The function $x \mapsto 4x+1$ *tends to* 5 *when* x *tends to* 1 *because* $|4x+1-5| = 4|x-1|$. *Taking* $\eta = \varepsilon/4$ *allows us to show that the property is verified.*



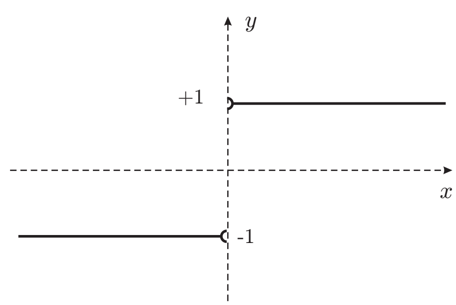
2. The function $x \mapsto \sin x/x$ *is not defined at* 0 *but trigonometry tell us that it tends to* 1 *when* x *tends to* 0.





(It is simply the geometric property that ensures that on a circle, the length of a chord and of the corresponding arc are equal up to a negligible function [1] *when the corresponding angle at the center tends to* 0.*)*

3. The function $x \mapsto x/|x|$ *is equal to 1 when* $x > 0$ *and* -1 *when* $x < 0$.

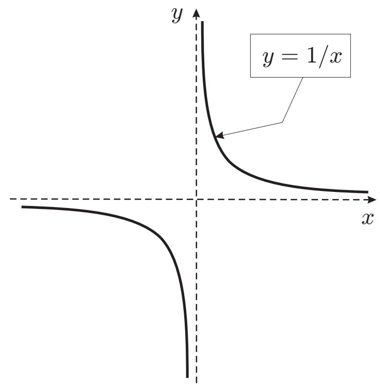


It does not have a limit when x *goes to* 0 *because for any* $\ell \in \mathbb{R}$, $|x/|x| - \ell| = |1-\ell|$ *for* $x > 0$ *and* $|x/|x| - \ell| = |1+\ell|$ *for* $x < 0$. *For any choice of* ℓ, *at least one of these quantities*

[1] the exact meaning of this will become clear during the course on asymptotic formulae

does not tend to 0, which shows that $x \mapsto x/|x|$ does not have a limit when x tends to 0. We will see later on that this function has a left-hand limit and a right-hand limit at 0 (these limits are respectively worth -1 and 1) that are not equal, thus preventing the definition of a limit from being satisfied.

4. The function $x \mapsto 1/x$ has values that become bigger and bigger in absolute value when x tends to 0.



It does not have a finite limit ℓ at $x = 0$.

5. The following example is fundamental. We consider the absolute value function $x \mapsto |x|$. The inequality

$$\Big| |x| - |a| \Big| \leq |x - a|$$

gives straight from the definition (by choosing $\eta = \varepsilon$) that, for any $a \in \mathbb{R}$, $|x|$ tends to $|a|$ when x tends to a.

Remark 2. *The previous examples lead to several remarks.*

1. The first function from the examples is defined at 0, but the second one is not. In fact, it is key to note that in the definition of the limit of a function, the function does not need to be defined at a.

2. The third function is bounded on a neighbourhood of 0 whereas the fourth one is not.

Finite limit when the variable goes to infinity

Definition 2.7. *Let f be a function defined over a neighbourhood of $+\infty$. We say that f tends to ℓ when x tends to $+\infty$ when it verifies*

$$\boxed{\forall \varepsilon > 0,\ \exists A \in \mathbb{R},\ \forall x \in \mathcal{D}_f, \quad x \in]A, +\infty[\implies |f(x) - \ell| \leq \varepsilon.}$$

In other words, f tends to ℓ when x tends to $+\infty$ if $|f(x) - \ell|$ can be made as small as we want as long as x is chosen to be big enough.

Definition 2.8. *Let f be a function defined over a neighbourhood of $-\infty$. The function will be said to tend to ℓ when x tends to $-\infty$ when it verifies*

$$\boxed{\forall \varepsilon > 0,\ \exists A \in \mathbb{R},\ \forall x \in \mathcal{D}_f, \quad x \in]-\infty, A[\implies |f(x) - \ell| \leq \varepsilon.}$$

Example 3. *Once again we give different examples to show different situations that we can come across.*

1. The function $x \mapsto 5 + 1/x$ *tends to* 5 *when* x *tends to* $+\infty$ *because as soon as* $x > A = 1/\varepsilon$, $|5 + 1/x - 5| = |1/x| \leq \varepsilon$.

2. The function $x \mapsto \sin x$ *does not have a limit when* x *goes to* $+\infty$ *because* $\sin(2k\pi) = 0$ *and* $\sin(\pi/2 + 2k\pi) = 1$ *for* $k \in \mathbb{N}$. *For all* $\ell \in \mathbb{R}$,

$$|\sin(2k\pi) - \ell| = |\ell| \;\; et \;\; |\sin(\pi/2 + 2k\pi) - \ell| = |1 - \ell| \,.$$

Once again, for any choice of ℓ, *at least of of the quantities does not go to 0. As a consequence,* $\sin x$ *cannot tend to a limit when* x *tends to* $+\infty$.

Limit uniqueness

The following theorem tells us that when a limit exists, it is unique. This justifies for example us saying "**the** limit of f at a" at the start of the chapter.

Theorem 2.9. *If* f *has two limits* ℓ *and* ℓ' *at point* $a \in \overline{\mathbb{R}}$, *then* $\ell = \ell'$. *We then denote*

$$\ell = \lim_{x \to a} f(x).$$

Proof We do the proof for $a = +\infty$ and $\ell \in \mathbb{R}$. The other cases are adaptations of this proof and we leave them as exercises for the reader.

$$\forall \varepsilon > 0 \begin{cases} \exists A_1 > 0, \ \forall x \in \mathcal{D}_f, \quad x > A_1 \implies \quad |f(x) - \ell| < \varepsilon \\ \exists A_2 > 0, \ \forall x \in \mathcal{D}_f, \quad x > A_2 \implies |f(x) - \ell'| < \varepsilon. \end{cases}$$

Let $\varepsilon > 0$ be fixed and $A_3 = \max(A_1, A_2)$. We fix $x \in \mathcal{D}_f$ such that $x > A_3$. This allows us to write

$$|\ell - \ell'| = |\ell - f(x) + f(x) - \ell'| \leq |\ell - f(x)| + |f(x) - \ell'| \leq 2\varepsilon,$$

which proves that $\ell = \ell'$. □

Remark 3. *We will sometimes also write* $\lim_{x \to a} f(x) = \ell$ *to say that* f *tends to* ℓ *when* x *tends to* a.

Infinite limit

Definition 2.10. *Let* f *be a function defined on a neighbourhood of* $a \in \mathbb{R}$. *We say that* f *tends to* $+\infty$ *when* x *tends to* a *when it verifies:*

$$\boxed{\forall A \in \mathbb{R}, \ \exists \eta > 0, \ \forall x \in \mathcal{D}_f, \quad |x - a| \leq \eta \implies f(x) \geq A.}$$

We then denote

$$\lim_{x \to a} f(x) = +\infty.$$

For a function f defined on a neighbourhood of $+\infty$, we say that f tends to $+\infty$ when x tends to $+\infty$ when it verifies

$$\boxed{\forall A \in \mathbb{R},\ \exists B \in \mathbb{R},\ \forall x \in \mathcal{D}_f, \quad x \geq B \implies f(x) \geq A.}$$

Similarly, for a function f defined on a neighbourhood of $-\infty$, we say that f tends to $+\infty$ when x tends to $-\infty$ when it verifies

$$\boxed{\forall A \in \mathbb{R},\ \exists B \in \mathbb{R},\ \forall x \in \mathcal{D}_f, \quad x \leq B \implies f(x) \geq A.}$$

We introduce the notation $\lim\limits_{x\to a} f(x) = +\infty$ for $a \in \overline{\mathbb{R}}$.

We now give a general definition to the notion of limit that fully exploits the notion of neighbourhood.

Definition 2.11. *Let f a function. Let $a \in \bar{\mathbb{R}}$. We suppose f to be defined on a neighbourhood of a (except perhaps at a).*
We say that the limit of the function f at a is $\ell \in \bar{\mathbb{R}}$ if

$$\boxed{\forall V \in \mathcal{V}(\ell), \exists W \in \mathcal{V}(a), \forall x \in \mathcal{D}_f \quad x \in W \Rightarrow f(x) \in V.}$$

2.2.2 Criteria for existence and properties

In this part we introduce tools that allow us to ensure the existence of limits or compute them.

Links with limits of sequences

The following theorem allows us to extend to limits of functions the properties of limits of sequences.

Theorem 2.12. *Let f be a function defined on a neighbourhood of $a \in \overline{\mathbb{R}}$. The following properties are equivalent:*

1. $\lim\limits_{x\to a} f(x) = \ell$;
2. *For any sequence $(u_n)_{n\in\mathbb{N}}$ belonging to $\mathcal{D}_f$ such that $\lim\limits_{n\to\infty} u_n = a$, we have $\lim\limits_{n\to\infty} f(u_n) = \ell$.*

Proof We give the proof for a and ℓ finite. We leave the adaptation of this proof to the cases where $a = \pm\infty$ or $\ell = \pm\infty$ as an exercise.

Let us show that (1) implies (2). Let $(u_n)_{n\in\mathbb{N}}$ belong to $\mathcal{D}_f$ such that $\lim\limits_{n\to\infty} u_n = a$. Let $\varepsilon > 0$. There exists $\eta > 0$ such that if $x \in \mathcal{D}_f$ verifies $|x - a| \leq \eta$ then $|f(x) - \ell| \leq \varepsilon$. As $\eta > 0$, $\exists n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we have that $|u_n - a| \leq \eta$. By choosing $x = u_n$, we then have that for any $n \geq n_0$, $|f(u_n) - \ell| \leq \varepsilon$. This means that $\lim\limits_{n\to\infty} f(u_n) = \ell$.

We will now show that (2) implies (1) and will prove this by proving that the converse is true. So let us suppose that f does not converge to ℓ when x tends to a. As a consequence there exists ε_0 such that, for any positive integer n, there exists (at least) one element of x in $\mathcal{D}_f$ verifiying $|x-a| \leq 1/(n+1)$ and $|f(x)-\ell| > \varepsilon_0$. For each n we set, u_n as one of these elements of x. We have thus built a sequence $(u_n)_{n\in\mathbb{N}}$ belonging to $\mathcal{D}_f$ that tends to a such that $(f(u_n))_{n\in\mathbb{N}}$ does not converge to ℓ. $\square$

The second point of Theorem 2.12 is called **sequential characterization** of limits. The previous theorem is also very useful to show that a function does not have at limit at a: we simply have to build either a sequence $(u_n)_{n\in\mathbb{N}}$ converging to a such that $(f(u_n))_{n\in\mathbb{N}}$ does not converge, or two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ both converging to a and such that $(f(u_n))_{n\in\mathbb{N}}$ and $(f(v_n))_{n\in\mathbb{N}}$ converge to two different limits (see part 3 of example 2).

Exercise 2. *Show that the function $x \mapsto |\sin(1/x)|$ does not have a limit at* 0.

Operations on limits

All the operations that we saw to be valid on sequence limits during the "Maths Algo" course extend to limits of functions.
Here is how we extend to limits of functions the results on limits of sequences (here we deal with the case of the sum of two functions):
Let f and g be two functions defined on a neighbourhood of $a \in \bar{\mathbb{R}}$ (except perhaps at a) that have a limit at a that is worth respectively $\ell \in \bar{\mathbb{R}}$ and $\ell' \in \bar{\mathbb{R}}$.
The function $f+g$ is defined on a neighbourhood of a.
Let $(u_n)_{n\in\mathbb{N}}$ be a sequence belonging to $\mathcal{D}_{f+g}$ and tending to a. According to Theorem 2.12, we have

$$\lim_{n\to+\infty} f(u_n) = \ell \quad \text{et} \qquad \lim_{n\to+\infty} g(u_n) = \ell'.$$

Thus, when the operation $\ell+\ell'$ is well defined, we have

$$\lim_{n\to+\infty} f(u_n) + g(u_n) = \lim_{n\to+\infty} (f+g)(u_n) = \ell + \ell'.$$

So, once again according to Theorem 2.12, we obtain

$$\lim_{n\to+\infty} (f+g)(x) = \ell + \ell'.$$

The case of the limit of a product of functions is dealt with in the same way.

For the inverse of a function : the function $\frac{1}{f}$ is defined on a neighbourhood of a (except perhaps at a) if the function f does not becomes zero on a neighbourhood of a (except perhaps at a).

The limits of the different of the quotient of two functions are obtained through the previous results.

In order to list the operations that we can perform on limits, let us first recall the determinate and indeterminate cases for operations on $\overline{\mathbb{R}}$. We give them in the following tables, and the indeterminate cases are noted as a question mark "?" . The first term of the operation is given by the row and the second term by the column; a and b indicate finite values; $\operatorname{sgn} a$ gives the sign of a (when $a \neq 0$).

$+$	$-\infty$	b	$+\infty$
$-\infty$	$-\infty$	$-\infty$	?
a	$-\infty$	$a+b$	$+\infty$
$+\infty$	?	$+\infty$	$+\infty$

$\times$	$-\infty$	$b \neq 0$	0	$+\infty$
$-\infty$	$+\infty$	$\operatorname{sgn}(-b)\infty$	?	$-\infty$
$a \neq 0$	$\operatorname{sgn}(-a)\infty$	ab	0	$\operatorname{sgn}(a)\infty$
0	?	0	0	?
$+\infty$	$-\infty$	$\operatorname{sgn}(b)\infty$	?	$+\infty$

From these basic cases, we deduce the following indeterminate forms:

$$\frac{\pm\infty}{\pm\infty} = ?, \quad \frac{0}{0} = ?, \quad +\infty - (+\infty) = ?, \quad -\infty - (-\infty) = ?,$$

$$1^{+\infty} = e^{+\infty \ln 1} = e^{\infty \times 0} = ?, \quad 1^{-\infty} = e^{-\infty \ln 1} = e^{-\infty \times 0} = ?,$$

$$+\infty^0 = e^{0 \ln(+\infty)} = e^{0 \times \infty} = ?, \quad 0^0 = e^{0 \ln(0)} = e^{0 \times (-\infty)} = ?$$

Theorem 2.13. *Let f and g be two functions defined on a neighbourhood of $a \in \overline{\mathbb{R}}$ that each tend to a finite or infinite limit when x tends to a. The following properties are verified:*

1. *if $f+g$ is defined on a neighbourhood of a and **if the result is not indeterminate**, then*
$$\lim_{x \to a}(f+g)(x) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) \ ;$$
2. *if fg is defined on a neighbourhood of a and **if the result is not indeterminate**, then*
$$\lim_{x \to a}(fg)(x) = \lim_{x \to a} f(x) \lim_{x \to a} g(x) \ ;$$

Theorem 2.14. *Let g be a function defined on a neighbourhood of $a \in \bar{\mathbb{R}}$, whose limit at a is $\ell' \in \bar{\mathbb{R}}$ and that is not worth zero on a neighbourhood of a (except perhaps at a). Then the function $x \mapsto \dfrac{1}{g(x)}$ is defined on a neighbourhood of a (except perhaps at a) and*

- *if $\ell' \in \mathbb{R}^*$, $\displaystyle\lim_{x \to a} \frac{1}{g(x)} = \frac{1}{\ell'}$,*
- *if $\ell' = -\infty$ or if $\ell' = +\infty$, $\displaystyle\lim_{x \to a} \frac{1}{g(x)} = 0$,*
- *if $\ell' = 0$ and if g is positive on a neighbourhood of a (except at a), $\displaystyle\lim_{x \to a} \frac{1}{g(x)} = +\infty$,*
- *if $\ell' = 0$ and if g is negative on a neighbourhood of a (except at a), $\displaystyle\lim_{x \to a} \frac{1}{g(x)} = -\infty$.*

Remark 6. *We deduce from this theorem and from the result on the limit of products of functions in the previous theorem, the results on quotients of functions.*

The following theorem allows us to compute the limit of the composition of functions

Theorem 2.15 (Composition). *Let g be a function that has a limit $\ell \in \overline{\mathbb{R}}$ when y tends to $b \in \overline{\mathbb{R}}$. Let f be a function defined on a neighbourhood $a \in \overline{\mathbb{R}}$ whose values are in $\mathcal{D}_g$ such that $\lim_{x \to a} f(x) = b$, then*

$$\lim_{x \to a} g(f(x)) = \ell.$$

Proof

$\square$

Exercise 3. *Determine, if it exists, the limit of the function $x \mapsto \ln\left(\frac{x-1}{x+1}\right)$ when x tends to $+\infty$.*

Let f and g be two functions defined on a neighbourhood of $a \in \overline{\mathbb{R}}$ that admit a limit at a. Suppose that the limit of f is non negative. To compute the limit at a of the function $x \mapsto f(x)^{g(x)}$, we have to use the fact that by definition

$$f(x)^{g(x)} := \exp(g(x)\ln(f(x))).$$

This allows us to clearly see any indeterminate forms and deal with them.

Exercise 4. *Determine (if it exists) the limit at $+\infty$ of*

$$\left(2 + \frac{1}{x}\right)^x.$$

Remark 4. *When f and g have respective limits at a, $\ell \in]0, +\infty[$ and $\ell' \in \mathbb{R}$, the limit at a of f^g is worth $\ell^{\ell'}$.*

Theorem 2.16 ("Squeeze theorem"). *Let f, g and h be three functions defined on a neighbourhood of $a \in \overline{\mathbb{R}}$. If f and h have the same limit ℓ at a and if there exists a neighbourhood V of a such that*

$$\forall x \in V \cap \mathcal{D}_g, \quad f(x) \leq g(x) \leq h(x),$$

then the limit of g at a is ℓ.

2.2.3 Right-hand and left-hand limit

Let us analyse the following examples:

- the function $x \mapsto x/|x|$ does not have a limit at 0 as we saw in part 3 of Example 2. However, if we consider the restriction of this function to the interval $]0, +\infty[$, this restriction stays equal to 1, so it tends to 1 when x tends to 0; similarly the restriction of $x \mapsto x/|x|$ to the interval $]-\infty, 0[$ stays equal to -1 so it tends to -1 when x tends to 0;

- in the same way, the restriction of $x \mapsto 1/x$ to $]0, +\infty[$ tends to $+\infty$ when x tends to 0 ; its restriction to $]-\infty, 0[$ tends to $-\infty$ when x tends to 0.

To deal with this kind of situation, we introduce the following notions.
In the rest of this part, a is defined as a point of $\mathbb{R}$.

Definition 2.17. *Let f be a function defined on a neighbourhood of a.*

- *We say that $\ell \in \overline{\mathbb{R}}$ is a* ***right-hand limit*** *for f at a if there exists $h > 0$ such that f is defined on $]a, a+h[$ and the restriction of f to $]a, a+h[$ has ℓ as a limit when x tends to a. We then denote*
$$\lim_{x \to a^+} f(x) = \ell.$$

- *We say that $\ell \in \overline{\mathbb{R}}$ is a* ***left-hand limit*** *for f at a if there exists $h > 0$ such that f is defined on $]a-h, a[$ and the restriction of f to $]a-h, a[$ has ℓ as a limit when x tends to a. We then denote*
$$\lim_{x \to a^-} f(x) = \ell.$$

Other notations are also used: $\lim\limits_{\substack{x \to a \\ x > a}} f(x) = \ell$ or even $f(a^+) = \ell$ for the right-hand limit and $\lim\limits_{\substack{x \to a \\ x < a}} f(x) = \ell$ or even $f(a^-) = \ell$ for the left-hand limit.

Example 4. *We have $\lim\limits_{x \to 0^\pm} x/|x| = \pm 1$ and $\lim\limits_{x \to 0^\pm} 1/x = \pm\infty$.*

The following proposition, whose proof is immediate from the definitions, leads us to a criteria that can be useful to determine whether a function has a limit or not.
In particular, it allows us to conclude on the existence or not of a limit at a, when the function at hand is defined through different expressions to the left and to the right of a.

Proposition 2.18. *Let f a function and $h > 0$ such that $]a-h, a[$ and $]a, a+h[$ are both included in $\mathcal{D}_f$. Then*
$$\lim_{x \to a} f(x) = \ell \in \overline{\mathbb{R}} \text{ if and only if } \lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = \ell\,.$$

Proof We deal with the case where $\ell \in \mathbb{R}$. The other cases are easy adaptations.

- Suppose that $\lim_{x\to a} f(x) = \ell$ then
$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall x \in \mathcal{D}_f, \quad |x-a| < \eta \Longrightarrow |f(x) - \ell| < \varepsilon.$$
But $0 < x - a < \eta \Longrightarrow |x-a| < \eta$ and $0 < a - x < \eta \Longrightarrow |x-a| < \eta$ so
$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall x \in \mathcal{D}_f, \quad 0 < x-a < \eta \Longrightarrow |f(x) - \ell| < \varepsilon,$$
and
$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall x \in \mathcal{D}_f, \quad 0 < a-x < \eta \Longrightarrow |f(x) - \ell| < \varepsilon$$
so $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^-} f(x) = \ell$

- Conversely, suppose that $\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = \ell$. Then
$$\forall \varepsilon > 0,\ \exists \eta_1 > 0,\ \forall x \in \mathcal{D}_f, \quad 0 < x-a < \eta_1 \Longrightarrow |f(x) - \ell| < \varepsilon,$$
and
$$\forall \varepsilon > 0,\ \exists \eta_2 > 0,\ \forall x \in \mathcal{D}_f, \quad 0 < a-x < \eta_2 \Longrightarrow |f(x) - \ell| < \varepsilon.$$
Let $\eta_3 = \min(\eta_1, \eta_2)$. If $|x-a| < \eta_3$ then $0 < x-a < \eta_3 < \eta_1$ or $0 < a-x < \eta_3 < \eta_2$ so $|f(x) - \ell| < \varepsilon$ and $\lim\limits_{x\to a} f(x) = \ell$.

□

Exercise 5. *Let f be the function defined as* $f(x) = \begin{cases} e^{-1/x} & si\ x > 0\,, \\ 0 & si\ x < 0\,. \end{cases}$
Does the function f have a limit at 0 ?

An important class of functions always has left-hand and right-hand limits: monotonic functions.

Proposition 2.19. *Let $f : I \to \mathbb{R}$ be a monotonic function on I an open interval of $\mathbb{R}$. Then for any $a \in I$, the function f has a left-hand and right-hand limit at a.*

Proof We give the proof for a non-decreasing function f and the right-hand limit at a in I. We use sequential characterisation of the limit (*i.e.* Proposition 2.12). Let $(x_n)_{n\in\mathbb{N}}$ be a non-increasing sequence of I such that for any $n, x_n > a$. As the function f is non descreasing, we have that $f(a) \le f(x_{n+1}) \le f(x_n)$, $\forall n \in \mathbb{N}$: the sequence $(f(x_n))_{n\in\mathbb{N}}$ is non decreasing and lower-bounded so it converges to a limit l. Furthermore, we have that $f(x_n) \ge l$ for any integer n. Let $(a_n)_{n\in\mathbb{N}}$ be a sequence of I such that $\lim_{n\to\infty} a_n = a$ and $a_n > a, \forall n \in \mathbb{N}$. For any n, we set $\varphi(n) := \inf\{p \in \mathbb{N},\ p > n,\ a_n \ge x_p\}$. This lower bound is always well-defined as the sequence $(x_n)_{n\in\mathbb{N}}$ converges to a. So for any n :
$$f(a_n) \ge f(x_{\varphi(n)}) \ge l.$$
Thus $|f(a_n) - l| = f(a_n) - l$. We set $\varepsilon > 0$: we know that there exists $N > 0$ such that $0 \le l - f(x_N) < \varepsilon$. The sequence $(a_n)_{n\in\mathbb{N}}$ converges to a so there exists $n_0 \in \mathbb{N}$ such that for any $n \ge n_0, a_n \le x_N$. Using the monotonicity of f, we obtain that $0 \le f(a_n) - l \le l - f(x_N) \le \varepsilon$ for any $n \ge n_0$: we therefore deduce that f has a right-hand limit at a. □

2.3 Continuity of a function

In all this section, I is a non empty interval containing more than one point.

2.3.1 Continuity at a point

Definition 2.20. *Let f be a function defined on I and let $a \in I$.*

The function f is continuous *at a if and only if $\lim_{x \to a} f(x) = f(a)$.*

The function f is discontinuous *at a if it is not continuous at a.*

In other words, f is continuous at a if f is defined at a and if we can swap the two symbols $\lim_{x \to a}$ and f:

$$\boxed{\lim_{x \to a} f(x) = f(\lim_{x \to a} x).}$$

Definition 2.21. *The function f is* right-continuous at *a if $\lim_{x \to a^+} f(x) = f(a)$.*

The function f is left-continuous at *a if $\lim_{x \to a^-} f(x) = f(a)$.*

Exercise 6. *Study the continuity of the following functions at a:*

1. $f : x \mapsto |x|$, $a \in \mathbb{R}$,
2. $g : x \mapsto \begin{cases} x/|x| & si\ x \in \mathbb{R}^* \\ 1 & si\ x = 0 \end{cases}$, $a = 0$,
3. $h : x \mapsto \begin{cases} e^{-1/x} & si\ x \in \mathbb{R}^* \\ 0 & si\ x = 0 \end{cases}$, $a = 0$.

2.3.2 Continuity on an interval

Definition 2.22. *The function f will be said to be* ***continuous on the interval*** *I if f continuous at every point $x \in I$.*

We denote $\mathcal{C}^0(I)$ ***the class of continuous functions*** *on I.*

Example 5. *The following examples illustrate different situations that we can encounter.*

1. Every polynomial function $x \mapsto p(x) = a_n x^n + \cdots + a_0$ is continuous on $\mathbb{R}$.

2. Every irreducible rational fraction $x \mapsto f(x) = p(x)/q(x)$ is continuous on any interval I that does not contain a pole of f.

3. The continuous extension of $x \mapsto \sin(x)/x$ to the point 0 is continuous over $\mathbb{R}$.

4.The function $x \mapsto \sqrt{x-1}$ is continuous on $[1, +\infty[$

Operations on continuous functions

The following theorem that establishes the operations that give us a continuous function from two continuous functions is a direct consequence of the theorem regarding operations on limits.

Theorem 2.23. *If two functions f and g are continuous on I, then*

1. *the functions $f + g$ and fg are continuous on I ;*
2. *the function f/g is continuous on I except at points x of I where $g(x) = 0$.*

Through the same method, we obtain the following theorem that states that the composition of two continuous functions is a continuous function.

Theorem 2.24. *If f is continuous on I and if g is continuous on an interval J with $f(I) \subset J$, then $x \mapsto g \circ f(x) = g(f(x))$ is continuous on I.*

Example 6. *As a direct consequence of the previous theorems and the continuity of the function $x \mapsto |x|$, we obtain that if f is a continuous function on I, then the function $|f|$, $f^+ = (|f| + f)/2$ and $f^- = (|f| - f)/2$ are continuous.*

2.4 Differentiability of a function

Once again, in this paragraph, I is a non empty interval containing more than one point and a is a point in I.

Just like the notion of continuity, the notion of differentiability is a local notion. We thus only need to give the definitions and the properties for functions that are defined on an interval. Throughout this part, $f : I \to \mathbb{R}$ is a real valued function on I.

2.4.1 Point-wise differentiability

Definition 2.25. *We call* ***Newton's difference quotient*** *of the function f at point a the function defined by*

$$\begin{aligned} t_a : I \setminus \{a\} &\to \mathbb{R} \\ x &\mapsto \frac{f(x) - f(a)}{x - a} \end{aligned}$$

Definition 2.26. *f is said to be* ***differentiable at*** *a if the function t_a has a finite limit at a and we denote*

$$\boxed{f'(a) = \lim_{x \to a} t_a(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}.}$$

Equivalently (through the change of variable $x = a + h$), f is differentiable at a if the limit of $\dfrac{f(a + h) - f(a)}{h}$ exists when h tends to 0.

Remark 7. *This definition rewrites as follows: f is differentiable at a if there exist a real number ℓ and a function ε, defined on a neighbourhood of 0, such that for a small enough h,*

$$f(a + h) = f(a) + \ell h + h\varepsilon(h), \quad \lim_{h \to 0} \varepsilon(h) = 0,$$

or equivalently such that for any x close enough to a,

$$f(x) = f(a) + \ell(x-a) + (x-a)\varepsilon(x-a), \quad \lim_{x\to a} \varepsilon(x-a) = 0.$$

The real number ℓ is then unique $\left(\text{it is worth } \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}\right)$. It is called the derivative of f at a and denoted $f'(a)$.

Definition 2.27. *The function f is said to be **right differentiable** (or **left differentiable**) **at** a if the limit of $\dfrac{f(x)-f(a)}{x-a}$ when x tends to a^+ (or a^-) exists and we denote*

$$f'_d(a) = \lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} \text{ and } f'_g(a) = \lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}.$$

Using the properties on limits introduced earlier on in the chapter, we immediately obtain the following result.

Proposition 2.28. *Let f be a function defined on I and $a \in I$. Then,*

1. *If a is not a boundary value of I, f is differentiable at a if and only if f is both left and right differentiable at a and $f'_d(a) = f'_g(a)$.*
2. *If a is the left-hand boundary of I, f is differentiable at a if and only if f is right differentiable at a.*
3. *If a is the right-hand boundary of I, f is differentiable at a if and only if f is left differentiable at a.*

Example 7.

1. *Let $n \in \mathbb{N}^*$ an integer, $a \in \mathbb{R}$ a real number. For any $x \neq a$,*

$$\frac{x^n - a^n}{x-a} = \sum_{k=0}^{n-1} a^{n-1-k}x^k \underset{x\to a}{\longrightarrow} \sum_{k=0}^{n-1} a^{n-1-k}a^k = na^{n-1}$$

   so $f : x \mapsto x^n$ is differentiable at a, with $f'(a) = na^{n-1}$.

2. *Study the differentiability of $x \mapsto \sqrt{x}$ for $x \in \mathbb{R}_+$.*
3. *Let $f : x \mapsto x^2 \sin\left(\dfrac{1}{x}\right)$. Check that f has a continuous extension at 0 and study the extension's differentiability at 0.*
4. *Let $f : \mathbb{R} \to \mathbb{R},\ x \mapsto |x|$. Study the differentiability of f at 0.*

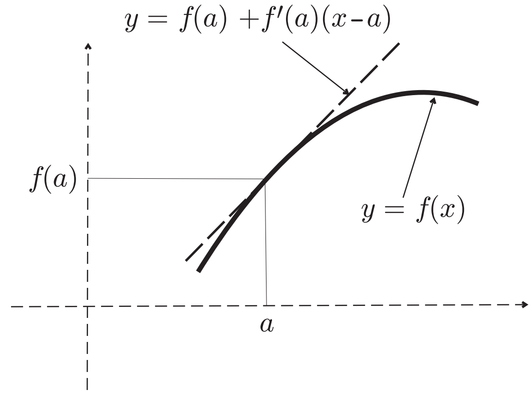


Figure 2.1: tangent to $\mathcal{C}_f$ at a

Link with the function's graph

Let f be a function defined on I and let $a \in I$. If f is differentiable at a, then we can write

$$f(x) = f(a) + f'(a)(x-a) + (x-a)\,\varepsilon(x), \quad \lim_{x\to a} \varepsilon(x) = 0.$$

The line defined by the equation $y = f(a) + f'(a)(x-a)$ is called the **tangent** to the curve of f at a. It is the "closest" line to the curve of f on a neighbourhood of a. Its slope is worth $f'(a)$ (Figure 2.1).

If f is right differentiable at a, the ray defined by the equation $y = f(a) + f'_d(a)(x-a),\ x > a$ is the **right semi-tangent** to the curve of f at a. If f is left differentiable at a, the ray defined by the equation $y = f(a) + f'_g(a)(x-a),\ x < a$ is the **left semi-tangent** to the curve of f at a. (Figure 2.2).

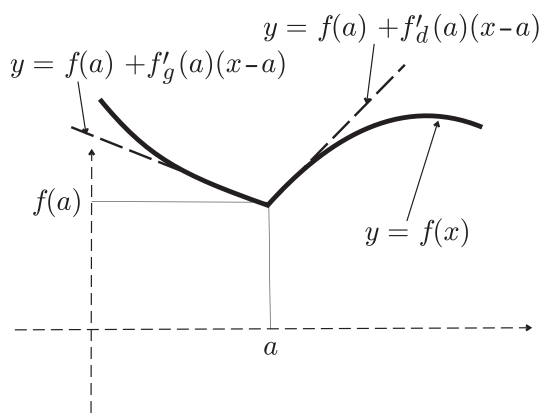
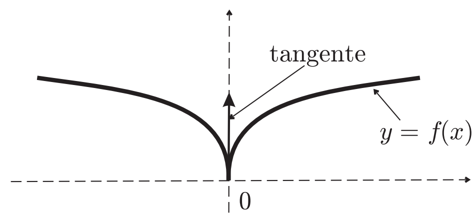


Figure 2.2: left and right semi-tangent to $\mathcal{C}_f$ at a

Generalisation. If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \pm\infty$, then f is not differentiable at a, but the vertical line defined by the equation $x = a$ is tangent to the curve of f at a. The vertical semi-tangents are defined in the same way.

Example 8. *Let $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto \sqrt{|x|}$. Let $x \neq 0$. We have $\frac{f(x) - f(0)}{x - 0} = \frac{1}{\sqrt{x}}$ if $x > 0$, $\frac{-1}{\sqrt{-x}}$ if $x < 0$, so $\lim_{x \to 0^+} \frac{f(x) - f(0)}{x} = +\infty$ and $\lim_{x \to 0^-} \frac{f(x) - f(0)}{x} = -\infty$: the vertical ray given by the equation $x = 0$, $y > 0$, is tangent to the curve of f at 0. At this point, the curve of f has a* ***cusp***.



Link with continuity

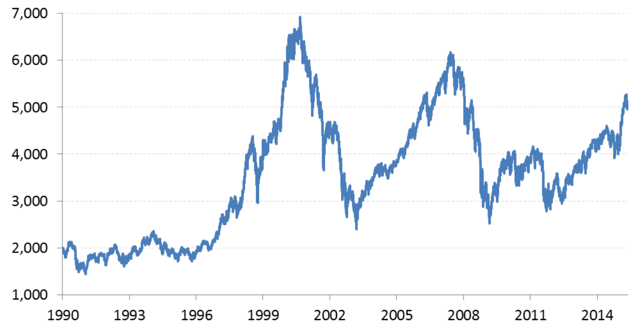
Proposition 2.29. *Let f be a function defined on the interval I and let $a \in I$. If f is differentiable at a then f is continuous at a.*

Attention. The converse is not true. Here is a counter example.

Example 9. *The absolute value function is continuous at 0, left and right differentiable at 0 but not differentiable at 0 (in this case, the left and right derivatives exist at 0 but they are not equal[2]).*

From the previous example, we could wish to conclude that for a continuous function $f : I \to \mathbb{R}$ the set of points where the function is not differentiable is "small", for example finite or countable. In reality many functions exist that are continuous on $\mathbb{R}$ but **never differentiable**. Of course, as they are not differentiable, they do not have any tangents and are thus very hard to draw. Here is an example of such a function found in finance where the price of a share (or more generally the price of a financial asset) is represented by a time dependent function that is continuous but never differentiable (see below some values from the CAC40).

[2] they are worth 1 and -1



2.4.2 Differentiability on a set

In the following, the sets I and J that we consider are unions of intervals.

Definition 2.30. *A function f is differentiable on I if f is differentiable at every point $x \in I$. In this case, the **derivative** of f on I is the function denoted f' defined by*

$$f' : \begin{array}{rcl} I & \to & \mathbb{R} \\ x & \mapsto & f'(x) \end{array}.$$

Operations on differentiable functions

Just like continuity, differentiability holds after operations such as addition, multiplication, division and composition. This is detailed in the following theorem.

Theorem 2.31. *Let f and g be two differentiable functions on I, λ a real number. Then*

1. *the function $\lambda f + g$ is differentiable on I, and*

$$\boxed{\forall x \in I,\ (\lambda f + g)'(x) = \lambda f'(x) + g'(x)},$$

2. *the product fg is differentiable on I, and*

$$\boxed{\forall x \in I,\ (fg)'(x) = f'(x)g(x) + f(x)g'(x),}$$

3. *the quotient $\dfrac{1}{f}$ is differentiable on $J = I \setminus \{x, f(x) = 0\}$ and*

$$\boxed{\forall x \in J,\ \left(\frac{1}{f}\right)'(x) = \frac{-f'(x)}{(f(x))^2}}$$

Proof Let $a \in I$. Let $x \neq a$.

1. We have

$$\frac{(\lambda f+g)(x)-(\lambda f+g)(a)}{x-a} = \lambda\frac{f(x)-f(a)}{x-a}+\frac{g(x)-g(a)}{x-a}$$

2. We have

$$\begin{aligned}\frac{(fg)(x)-(fg)(a)}{x-a} &= \frac{f(x)g(x)-f(a)g(x)+f(a)g(x)-f(a)g(a)}{x-a}\\ &= \frac{f(x)-f(a)}{x-a}g(x)+f(a)\frac{g(x)-g(a)}{x-a}.\end{aligned}$$

3. Let $a \in I$ such that $f(a) \neq 0$. Let $x \neq a$.

$$\frac{\frac{1}{f(x)}-\frac{1}{f(a)}}{x-a} = \frac{1}{f(x)f(a)}\frac{f(a)-f(x)}{x-a}$$

$\square$

The following theorem, on the derivation of composite functions, is called called the "chain rule".

Theorem 2.32. *If f is differentiable on I and if g is differentiable on an interval J such that $f(I) \subset J$, then $g \circ f$ is differentiable on I, and we have*

$$\boxed{\forall x \in I,\ (g\circ f)'(x) = g'(f(x))f'(x).}$$

Proof

$\square$

Example 11. *Study the differentiability and give the derivatives of the following functions:*

$$x \mapsto e^{x^2}, \qquad x \mapsto \ln(1+e^{x^2}), \qquad x \mapsto \cos^n x,\ n \in \mathbb{N}^*.$$

Theorem 2.33. *Let f and g be two functions that are differentiable on I. Then the quotient f/g is differentiable except at the points of I where g is worth 0, and we have*

$$\boxed{\forall x \in I \text{ such that } g(x) \neq 0, \quad \left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}.}$$

Proof The result is obtained by combining the previously obtained result on the product and the inverse. $\square$

2.4.3 Higher order derivatives

Definition 2.34. *Let $f : I \to \mathbb{R}$ be a differentiable function on a neighbourhood of $a \in I$.*

1. *If the derivative f', defined on a neighbourhood of a, is differentiable at a, then f is said to be* ***twice differentiable*** *at a, and the real number $(f')'(a)$ is denoted $f''(a)$.*

2. *Let $p \in \mathbb{N}$. We define $f^{(p)}(a)$ by induction. We set*

$$f^{(0)}(a) = f(a), f^{(1)}(a) = f'(a).$$

   If $f^{(p)}$ is defined on a neighbourhood of a and is differentiable at a, its derivative at a is denoted $f^{(p+1)}(a)$.

3. *If $f^{(p)}(a)$ exists, f is said to be* ***p times differentiable at a and $f^{(p)}(a)$ is the p'th derivative*** *or* ***derivative of order p*** *of f at a.*

4. *If $f^{(p)}(a)$ exists for all $a \in I$, f is said to be p times differentiable on I and we call p'th derivative or derivative of order p of f on I the function*

$$f^{(m)} : \begin{array}{rcl} I & \to & \mathbb{R} \\ x & \mapsto & f^{(p)}(x) \end{array}.$$

5. *We say that f belongs to* **the $\mathcal{C}^p$ space** *on I if $f^{(p)}$ exists and is continuous on I. The set of functions belonging to the $\mathcal{C}^p$ space on I is denoted $\mathcal{C}^p(I)$.*

6. *We say that f belongs to* **the $\mathcal{C}^\infty$ space** *on I if f belongs to the $\mathcal{C}^p$ space on I for all $p \in \mathbb{N}$.*

Operations

Theorem 2.35. *Let $p \in \mathbb{N}$. Let f and g two functions belonging to the $\mathcal{C}^p$ space on I, h a function belonging to the $\mathcal{C}^p$ space on J such that $f(I) \subset J$, λ a real number. Then*

1. *the function $\lambda f + g$ belongs to the $\mathcal{C}^p$ space on I,*

2. *the product fg belongs to the $\mathcal{C}^p$ space on I, and $(fg)^{(p)} = \sum_{k=0}^{p} C_p^k f^{(k)} g^{(p-k)}$ (Leibniz formula),*

3. *the composite function $h \circ f$ belongs to the $\mathcal{C}^p$ space on I,*

4. *if g is non zero on I, f/g belongs to the $\mathcal{C}^p$ space on I.*

Proof All of these properties can be shown through an induction on p.
1. This result is immediate.
2. The result is known for $p = 0$ (continuity). Suppose that the result is true for p. Let $f, g \in \mathcal{C}^{p+1}(I)$. Through our induction hypothesis, fg belongs to the $\mathcal{C}^p$ space on I and its p'th order derivative is given by $(fg)^{(p)} = \sum_{k=0}^{p} C_p^k f^{(k)} g^{(p-k)}$. The function f belongs to the $\mathcal{C}^{p+1}$ space so for any $0 \leq k \leq p$, $f^{(k)}$ belongs to the $\mathcal{C}^1$ space at least. Similarly, $g^{(p-k)}$ belongs at least to the $\mathcal{C}^1$ space. Through *1.*, $(fg)^{(p)} = \sum_{k=0}^{p} C_p^k f^{(k)} g^{(p-k)}$ thus belongs to the $\mathcal{C}^1$ space. Therefore, fg belongs to the $\mathcal{C}^{p+1}$ space on I.
Furthermore,

$$\begin{aligned}
(fg)^{(p+1)} &= \left(\sum_{k=0}^{p} C_p^k f^{(k)} g^{(p-k)} \right)' \\
&= \sum_{k=0}^{p} C_p^k f^{(k+1)} g^{(p-k)} + \sum_{k=0}^{p} C_p^k f^{(k)} g^{(p+1-k)} \\
&= f^{(p+1)} g + \sum_{k=1}^{p} \left(C_p^{k-1} + C_p^k \right) f^{(k)} g^{(p-k)} + f g^{(p+1)} \\
&= f^{(p+1)} g + \sum_{k=1}^{p} C_{p+1}^k f^{(k)} g^{(p+1-k)} + f g^{(p+1)} \\
&= \sum_{k=0}^{p+1} C_{p+1}^k f^{(k)} g^{(p+1-k)}.
\end{aligned}$$

So the result holds for $p+1$.
3. The result is known for $p=0$ (continuity). Suppose the result to be true for a general p. Let f, h belong to the $\mathcal{C}^{p+1}$ space on I and J respectively. Then $h \circ f$ is differentiable and $(h \circ f)' = h' \circ f \cdot f'$. As h' and f belong to the $\mathcal{C}^p$ space, through the induction hypothesis $h' \circ f$ belongs to the $\mathcal{C}^p$ space. Furthermore f' belongs to the $\mathcal{C}^p$ space so according to 2., we deduce that $(h \circ f)'$ belongs to the $\mathcal{C}^p$ space, and $h \circ f$ to the $\mathcal{C}^{p+1}$ space.
4. We apply *3.* to $h \circ g$ with $h : x \mapsto 1/x$ in order to express the derivative of $\frac{1}{g}$ and then we apply *2.* to the product of $\frac{1}{g}$ and f. □

Example 10. *Regularity of standard functions.*

1. *Polynomials belong to the* $\mathcal{C}^\infty$ *space on* $\mathbb{R}$.
2. *Rational fractions belong to the* $\mathcal{C}^\infty$ *space on their definition space (*$\mathbb{R}$ *without the poles).*
3. *The exponential function* exp *belongs to the* $\mathcal{C}^\infty$ *space on* $\mathbb{R}$.
4. *The logarithm function* ln *belongs to the* $\mathcal{C}^\infty$ *space on* $]0, +\infty[$.
5. *For all* $a \in \mathbb{R}$, $x \mapsto x^a$ *belongs to the* $\mathcal{C}^\infty$ *space on* $]0, +\infty[$.
6. *The trigonometric functions* cos*,* sin *belong to the* $\mathcal{C}^\infty$ *space on* $\mathbb{R}$.

2.5 Recap quiz

1. We consider the following subsets of $\mathbb{R}$:

$$]-3;+\infty[\ ,\quad [1;2]\cup[3;5[\ ,\quad \mathbb{R}^*\ ,\quad]-\infty;17[.$$

   List the ones that are intervals of $\mathbb{R}$.

2. Let $a \in \mathbb{R}$. Give a neighbourhood of a.

3. Give a neighbourhood of $+\infty$, and of $-\infty$.

4. Let f a real function. Write the following statement with quantifiers :
   "f tends to b when x tends to a"
   in the following cases :

   (a) $a \in \mathbb{R}, b \in \mathbb{R}$.
   (b) $a = +\infty, b \in \mathbb{R}$.
   (c) $a = +\infty, b = -\infty$.

5. Give a limit characterisation using numerical sequences. Deduce that the function $x \mapsto \cos x$ does not have a limit at $+\infty$.

6. State five indeterminate cases for limit computations.

7. Let f be a function defined on I. Let $a \in I$. Write the following sentence using quantifiers: "f is continuous at a".

8. Let f be a function defined on an interval I. Let $a \in I$. What does it mean for f to be differentiable at a?

9. What link is there between "f is continuous at a" and "f is differentiable at a". Give an example where the converse is not true.

10. Give, when it exists, the derivative of the sum, the product, the quotient and the composite of two differentiable functions.

11. Study the continuity, the differentiability and give the derivatives of the following functions:

    (a) $x \longmapsto f(x) = \cos(2x+1) - \frac{1}{x}$.

    (b) $x \longmapsto g(x) = \dfrac{\ln x}{\mathrm{ch} x}$.

    (c) $x \longmapsto h(x) = \sqrt{x^2 - 4}$.

    (d) $x \longmapsto k(x) = x^2 e^{3x}$.

    (e) $x \longmapsto l(x) = x^2 \sin(x^2)$.

12. Compute the following limits:

$$\lim_{x \to 0} \frac{\ln(1+x)}{x} \quad , \quad \lim_{x \to 0} \frac{\sin x}{x}.$$

13. Let F a differentiable function on $\mathbb{R}$. Compute $\displaystyle\lim_{x \to 2} \frac{F(x^3) - F(8)}{x - 2}$.

14. What does it mean when f belongs to the $\mathcal{C}^4$ space? What does it mean when f belongs to the $\mathcal{C}^\infty$ space?

15. Fill in the following table with the derivatives of standard functions and specify the definition, continuity and differentiability domains.

Definition domain	Continuity domain	Differentiability domain	$f(x)$	$f'(x)$
			$a\ ,\ a \in \mathbb{R}$	
			$x^n\ ,\ n \in \mathbb{N}^*$	
			$x^{-n}\ ,\ n \in \mathbb{N}^*$	
			$\frac{1}{x}$	
			$\sqrt{x}$	
			$\ln\ x$	
			e^x	
			$\sin\ x$	
			$\cos\ x$	
			$\tan\ x$	
			$\text{Arctan}\ x$	

Do not hesitate to fill it in further with any new functions you encounter.

Definition domain	Continuity domain	Differentiability domain	$f(x)$	$f'(x)$
			$\text{sh } x$	
			$\text{ch } x$	
			$\text{th } x$	
			$\text{Arcsin } x$	
			$\text{Arccos } x$	
			$\text{Argsh } x$	
			$\text{Argch } x$	
			$\text{Argth } x$	
			$x^\alpha \ , \ \alpha \in \mathbb{R}^*$	
			$\alpha^x \ , \ \alpha \in \mathbb{R}_+^*$	

16. Let u a differentiable function on an interval I of $\mathbb{R}$. Fill in the following table (we suppose that all the composite functions are differentiable on I):

f	f'	f	f'	f	f'
$u^n\ ,\ n \in \mathbb{N}^*$		$\sin\ u$		Arcsin u	
$u^{-n}\ ,\ n \in \mathbb{N}^*$		$\cos\ u$		Arccos u	
$\frac{1}{u}$		$\tan\ u$		Arctan u	
$\sqrt{u}$		sh u		Argsh u	
$\ln\ u$		ch u		Argch u	
e^u		th u		Argth u	

Chapter 3

Fundamental calculation techniques for analysis

3.1 Curve sketching

The aim of this section is to discuss methods that allow us to study a function and then sketch its curve, compute its extrema, show the existence of zeros...

3.1.1 Defining the study space

Definition domain, parity, periodicity

Each function study should begin with a study of its definition domain.

Definition 3.1. *Let $U \subset \mathbb{R}$ and $f : U \to \mathbb{R}$ a function. The definition domain of f, $\mathcal{D}_f$, is the largest subset of U on which f is defined.*

Example 12.

1. *The sine and cosine functions, all polynomial functions, and the exponential function are defined on $\mathbb{R}$.*

2. *The logarithmic function is defined on $\mathbb{R}_+^*$*

3. *As we saw earlier in the course, (Chapiter 3, Definition 3.38), by convention the definition domain of a rational fraction is given by the definition domain of its irreducible form. Thus the function $f : x \mapsto \dfrac{(x-1)}{x^2-3x+2}$ is defined on $\mathbb{R}\backslash\{2\}$ because its irreducible form is: $f(x) = 1/(x-2)$ (indeed $x^2 - 3x + 2 = (x-1)(x-2)$).*

4. *We consider the function $\tan : \mathbb{R} \to \mathbb{R}$ defined by $\tan(x) = \dfrac{\sin(x)}{\cos(x)}$: compute its definition domain.*

Some functions have parity or periodicity properties that allow us to reduce the study space.

Definition 3.2. *Let $f : U \to \mathbb{R}$.*

1. *We say that f is even if its definition domain is symmetrical with respect to 0 (i.e. if $x \in \mathcal{D}_f$ then $-x \in \mathcal{D}_f$) and if for any $x \in \mathcal{D}_f$, we have $f(-x) = f(x)$.*
2. *We say that f is odd if its definition domain is symmetrical with respect to 0 (i.e. if $x \in \mathcal{D}_f$ then $-x \in \mathcal{D}_f$) and if for any $x \in \mathcal{D}_f$, we have $f(-x) = -f(x)$.*

When a function is odd or even, studying half of its definition domain $\mathcal{D}_f \cap \mathbb{R}_+$ is enough for us to know its full behaviour. Geometrically, an even function is symmetrical with respect to the y axis and an odd function is symmetrical with respect to the x axis.

Example 13.

1. *The cosine function is even, the sine function is odd.*
2. *Let $k \in \mathbb{N}$: the function $x \mapsto x^{2k}$ is even and the function $x \mapsto x^{2k+1}$ is odd.*
3. *Let $f : \mathbb{R} \to \mathbb{R}$. We set for $x \in \mathbb{R}$, as we saw in the "Maths Algo" course, $g(x) = \dfrac{f(x) + f(-x)}{2}$ and $h(x) = \dfrac{f(x) - f(-x)}{2}$. g is the even part of f and h is the odd part of f. Note that $f = g + h$.*
4. *The function $ch : x \mapsto \dfrac{\exp(x) + \exp(-x)}{2}$ is even. The function $sh : x \mapsto \dfrac{\exp(x) - \exp(-x)}{2}$ is odd. They are the even and odd parts of the exponential function.*

Definition 3.3. *Let $f : \mathbb{R} \to \mathbb{R}$. We say that f is T-periodic ($T > 0$) if for any $x \in \mathcal{D}_f$, we have $x + T \in \mathcal{D}_f$ and $f(x + T) = f(x)$.*

When a function f is T-periodic, studying it on an T length interval is enough to know its full behaviour.

Example 14.

1. *The cosine and sine functions are 2π-periodic: we can study them on a 2π length interval. As they are also respectively even and odd, we deduce that we can just study then on $[0, \pi]$.*
2. *Give the study domain that allows us to know the full behaviour of the* $\tan$ *function.*

3.1.2 Variation table

Studying the variations of a function is closely linked to studying the sign of its derivative. Before computing the variation table of a function f, it is thus necessary to find the set $\mathcal{D}_d \subset \mathcal{D}_f$ of points on which f is differentiable.

Link between monotonicity and derivative

Let $I \subset \mathbb{R}$ an interval of $\mathbb{R}$.

Definition 3.4.

1. *f is non decreasing on I if for any $x, y \in I$ $x \leq y \Rightarrow f(x) \leq f(y)$.*
2. *f is non increasing on I if for any $x, y \in I$ $x \leq y \Rightarrow f(x) \geq f(y)$.*

Replacing the inequality signs by strict inequalities leads to an increasing function in the first case, and a decreasing function in the second case. Thus this gives strict monotonicity. Another way to introduce this to to add the property $x \neq y \Rightarrow f(x) \neq f(y)$". In other words, f is strictly monotonous if and only if it is monotonous AND injective.

Example 15.

1. *The function $x \mapsto x^3$ is non decreasing on $\mathbb{R}$.*
2. *The function $x \mapsto x^2$ is non decreasing on $\mathbb{R}_+$ and non increasing on $\mathbb{R}_-$.*

In general it is quite hard to establish a function's monotonicity straight from its definition. It is often easier to use a criteria related to its derivative. We start off with the following observation.

Remark 8.

1. *If the function $f : I \to \mathbb{R}$ is non decreasing and differentiable on I then $f'(x) \geq 0$ for any $x \in I$.*
2. *If the function $f : I \to \mathbb{R}$ is non increasing and differentiable on I then $f'(x) \leq 0$ for any $x \in I$.*

These properties can be shown straight from the definitions. The converse, that will be useful to study a function's monotonicity is harder to show. Its proof uses the mean value theorem that will be proven at the end of this course.

Proposition 3.5. *Let $I = [a, b]$ and $f : [a, b] \to \mathbb{R}$ a continuous and differentiable function on $[a, b]$. If $f'(x) \geq 0$ for any $x \in]a, b[$ then f is non decreasing on $[a, b]$. If $f'(x) > 0$ for any $x \in]a, b[$ then f is increasing on $[a, b]$.*

After replacing f by $-f$, we obtain a similar criteria for non increasing and decreasing functions.

Example 16.

1. *The function* $x \mapsto \sin(x)$ *is increasing on* $\left[0, \frac{\pi}{2}\right]$.

2. *The function* $x \mapsto x^3 + x$ *is increasing on* $\mathbb{R}$.

3. *The function* $f : \mathbb{R} \to \mathbb{R}$ *such that* $f(x) = 0$ *if* $x < 0$ *and* $f(x) = x^2$ *if* $x \geq 0$ *is non decreasing on* $\mathbb{R}$ *and increasing on* $\mathbb{R}_+$.

4. *The function* $x \mapsto x^3$ *is increasing on* $\mathbb{R}$ *(even though its derivative is worth* 0 *at* 0*!).*

Explicitly computing the derivative of a function f also gives some local information allowing us to sketch the outline of the curve of f that we denote as $\mathcal{C}_f$ in the following.

Definition 3.6. *Let* $f :]a, b[\to \mathbb{R}$ *a differentiable function (on* $]a, b[$*) and let* $c \in]a, b[$*. The line given by the equation* $y = f(c) + f'(c)(x - c)$ *is called the tangent to the curve* $\mathcal{C}_f$ *of* f *at* c*.*

We deduce from Remark 7 that the tangent to $\mathcal{C}_f$ at a is the linear function that best approaches $\mathcal{C}_f$ on a neighbourhood of a.
To study the position of the tangent with respect to $\mathcal{C}_f$ at c, we introduce the auxiliary function $g(x) = f(x) - f(c) - f'(c)(x - c)$ and we study its sign. We note that the tangent to $\mathcal{C}_f$ at a point c such that $f'(c) = 0$ (c is then said to be a critical point) is very easy to sketch as it is a horizontal line given by the equation $y = f(c)$.

Behaviour at the bounds of the definition domain

In the following, we denote as $\mathcal{C}_f$ the curve representing the function f,

$$\mathcal{C}_f = \{(x, f(x)) \,|\, x \in \mathcal{D}_f\}.$$

Definition 3.7. *When one of the two coordinates* x *or* $y = f(x)$ *tends to* $\pm\infty$*,* $\mathcal{C}_f$ *is said to have an* ***infinite branch****.*

We often wish to determine the behaviour of this infinite branch. In particular, we may wish to determine whether $\mathcal{C}_f$ is "similar" to a simpler curve? To do this, we first define the notion of asymptotic curve.

Definition 3.8. *Let* f *and* g *two functions defined on a neighbourhood of* a *(finite or* $\pm\infty$*). We say that* $\mathcal{C}_f$ *and* $\mathcal{C}_g$ *are* ***asymptotes*** *(or that the asymptote of* $\mathcal{C}_f$ *is* $\mathcal{C}_g$*) on a neighbourhood of* a *if* $\lim\limits_{x \to a}(f(x) - g(x)) = 0$.

Once we know that two curves are asymptotes on a neighbourhood of a (finite or $\pm\infty$), we may often wonder what their relative position is on a neighbourhood of a. To answer this question, we can study the sign of $f - g$ on a neighbourhood of a.

Definition 3.9. *1. If $f(x) - g(x) \geq 0$ on a neighbourhood of a, then $\mathcal{C}_f$ is* ***above*** *$\mathcal{C}_g$ at a.*

2. If $f(x) - g(x) \leq 0$ on a neighbourhood of a, then $\mathcal{C}_f$ is ***below*** *$\mathcal{C}_g$ at a.*

Remark 5.
- *If $\lim\limits_{x \to a} f(x) = \pm\infty$ then the line defined by the equation $x = a$ is an asymptote to the curve of f on a neighbourhood of a.*
- *If $\lim\limits_{x \to \pm\infty} f(x) = b \in \mathbb{R}$ then the line defined by the equation $y = b$ is an asymptote to the curve of f on a neighbourhood of a.*

An example of a curve

Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^3 - x$. The function f is defined and differentiable on $\mathbb{R}$ with $f'(x) = 3x^2 - 1$. Furthermore f is an odd function: it suffices to study f on $\mathbb{R}^+$. It is clear that $f'(x) < 0$ if $x \in [0, 1/\sqrt{3}[$ and $f'(x) > 0$ if $x > 1/\sqrt{3}$. We have that $f(0) = 0$ and $\lim\limits_{x \to \infty} f(x) = +\infty$. The variation table is given by

To sketch the curve of f, we specify its behaviour near certain points. At $x = 1/\sqrt{3}$, the derivative cancels out and the line given by the equation $y = f(1/\sqrt{3})$ is tangent to $\mathcal{C}_f$ and, furthermore, $f(x) \geq f(1/\sqrt{3})$ for any $x \geq 0$. We have two other notable points: 0 and 1, where f cancels out. At 0, the line given by the equation $y = -x$ is tangent to $\mathcal{C}_f$. At 1, the line given by $y = 2(x - 1)$ is tangent to $\mathcal{C}_f$. We thus have the graph:

3.1.3 Applications

Comparing, Upper bounding, Lower bounding

Being able to study the variations of a function can allow us to compare two functions or even allow us to upper/lower bound a function.

Example 17.

1. *Let $g : \mathbb{R} \to \mathbb{R}$ defined as $g(x) = \exp(x) - x - 1$. The function g is defined and differentiable on $\mathbb{R}$. We have that $g'(x) = \exp(x) - 1$ so $g'(x) < 0$ when $x < 0$ and $g'(x) > 0$ when $x > 0$. The function g reaches its minimum at $x = 0$ so $g(x) \geq g(0) = 0$. Thus for any $x \in \mathbb{R}$, we have $\exp(x) \geq x + 1$.*

2. *Let $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = \exp(x) - x^2$. The function h is defined and differentiable on $\mathbb{R}$. We have that $h'(x) = \exp(x) - 2x$ and $h''(x) = \exp(x) - 2$. We also know that $\exp(x) \geq x + 1$ so if $x \geq 1$, we have $h''(x) \geq 0$. The function h' is thus non decreasing on $[1, +\infty[$ and $h'(x) \geq h'(1) = e - 2 > 0$ so h is non decreasing on $[1, +\infty[$. We thus have $h(x) \geq h(1) = e - 1 > 0$ if $x \geq 1$. From this we deduce a first result that involves growth comparison. It is a "croissance comparée" theorem in french and so we will say that is is a growth comparison theorem.*

$$\lim_{x\to\infty} \frac{\exp(x)}{x} \geq \lim_{x\to\infty} x = +\infty.$$

In the following example, we compare the growth of $f : x \mapsto \ln(x + 1)$ to that of polynomial functions.

Example 18.

1. *Let $g : x \mapsto \ln(1 + x) - x$. The function g is defined and differentiable on $]-1, +\infty[$ with $g'(x) = 1/(1 + x) - 1$. So $g'(x) > 0$ if $x \in]-1, 0[$ and $g'(x) < 0$ otherwise. So $g(x) \leq g(0) = 0$ for any $x \in]-1, +\infty[$. Thus $\ln(1 + x) \leq x$ for any $x > -1$.*

2. *Let $h : x \mapsto \ln(1+x) - x + x^2/2$. We have that $\mathcal{D}_h = \mathcal{D}_g$ and $h'(x) = 1/(1+x) - 1 + x$ and $h''(x) = -1/(1 + x)^2 + 1$ so h' is non increasing on $]-1, 0[$ and non decreasing on $]0 + \infty[$. Furthermore $h'(0) = 0$ so $h'(x) \geq h'(0) = 0$. Thus $h(x) \geq h(0)$ if $x \geq 0$. We deduce that*

$$x - \frac{x^2}{2} \leq \ln(1 + x) \leq x, \quad \forall x \geq 0.$$

3.2 Standard functions

3.2.1 Logarithmic functions, exponential functions

The definition of the natural logarithm (or Napierian logarithm, named after John Napier) is based on the following theorem that will be accepted as true throughout this course.

Theorem 3.10. *Let I an interval of $\mathbb{R}$ and $f : I \to \mathbb{R}$ a continuous function on I. Then f has a primitive: there exists a function $F : I \to \mathbb{R}$ such that $F' = f$.*

The function F is called a primitive of f. It is defined up to a constant, which means that if F is a primitive of f, then $x \mapsto F(x) + a$ (with a any real number) is too.

Definition 3.11. *The function $x \mapsto 1/x$ is continuous on $\mathbb{R}_+^*$. We call natural logarithm the primitive of $x \mapsto 1/x$ which is equal to 0 at $x = 1$.*

We have the following properties:

Proposition 3.12.

1. *The logarithm function* $\ln$ *is increasing on* $]0, +\infty[$.
2. *The logarithm function* $\ln$ *is differentiable (and even belongs to the $\mathcal{C}^\infty$ space) on* $]0, +\infty[$ *and for all $x > 0$,* $(\ln)'(x) = 1/x$.
3. *For all $a, b > 0$,* $\ln(ab) = \ln(a) + \ln(b)$. *We deduce that* $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b)$.
4. *For all $a > 0$ and $n \in \mathbb{N}^*$,* $\ln(a^n) = n\ln(a)$ *and* $\ln(a^{\frac{1}{n}}) = \frac{1}{n}\ln(a)$.
5. $\lim_{x\to\infty} \ln(x) = +\infty$, $\lim_{x\to 0^+} \ln(x) = -\infty$.
6. $\lim_{x\to 0} \frac{\ln(1+x)}{x} = 1$.

The definition of the exponential function is based on the following property.

Proposition 3.13. *Let $f : I \to \mathbb{R}$ a continuous and increasing function: then f is a bijection from I to $J = f(I)$. Furthermore, there exists a unique function, called inverse function and denoted f^{-1} such that for any $x \in I$ we have $f^{-1}(f(x)) = x$ and for any $y \in J$, we have $f(f^{-1}(y)) = y$.*

Note that we will reuse the previous result in Chapter 8 (in Section 4.3.2).

Definition 3.14. *The logarithm function is a bijection from $\mathbb{R}_+^*$ to $\mathbb{R}$. We denote* $\exp$ *its inverse function defined on $\mathbb{R}$ and valued in $\mathbb{R}_+^*$. For any $x > 0$, we have* $\exp(\ln(x)) = x$ *and for any $y \in \mathbb{R}$, we have* $\ln(\exp(y)) = y$.

The exponential function verifies the following properties. The continuity and differentiability properties will be proven further on.

Proposition 3.15.

1. *The function* $\exp$ *is defined, continuous and increasing on* $\mathbb{R}$
2. *The function* $\exp$ *is differentiable (and even belongs to the $\mathcal{C}^\infty$ space) on $\mathbb{R}$ and* $\exp'(x) = \exp(x)$ *for any $x \in \mathbb{R}$.*
3. *For any $a, b \in \mathbb{R}$,* $\exp(a + b) = \exp(a)\exp(b)$
4. $\lim_{x\to+\infty} \exp(x) = +\infty$, $\quad \lim_{x\to-\infty} \exp(x) = 0$

5. $\lim_{x \to 0} \dfrac{\exp(x) - 1}{x} = \exp'(0) = 1.$

We can also define the exp function as the solution $u : \mathbb{R} \to \mathbb{R}$ to the problem

$$u'(x) = u(x),\ \forall x \in I, \quad u(0) = 1.$$

Proposition 3.16 (growth comparison)**.** *Let $\alpha > 0$ and $\beta > 0$, we have that:*

1. $\lim_{x \to +\infty} \dfrac{\ln x}{x} = 0^+,$

2. $\lim_{x \to +\infty} \dfrac{\ln x}{x^\alpha} = 0^+,$

3. $\lim_{x \to +\infty} \dfrac{(\ln x)^\beta}{x^\alpha} = 0^+,$

4. $\lim_{x \to +\infty} \dfrac{e^x}{x^\alpha} = +\infty$

5. $\lim_{x \to +\infty} \dfrac{e^{\beta x}}{x^\alpha} = +\infty,$

6. $\lim_{x \to 0^+} x \ln x = 0^-,$

7. $\lim_{x \to 0^+} x^\alpha \ln x = 0^-.$

Proof

- For any real $x \geq 4$, let us set $d(x) = \ln x - \sqrt{x}$.
  The function d is defined, continuous and differentiable on $[4, +\infty[$ as the subtraction of two functions that are defined, continuous and differentiable on $[4, +\infty[$.
  For any real $x \geq 4$, $\sqrt{x} \geq 2$ so

$$d'(x) = \frac{1}{x} - \frac{1}{2\sqrt{x}} = \frac{2 - \sqrt{x}}{2x} \leq 0.$$

  The function d is thus non increasing on $[4, +\infty[$ and for any real $x \geq 4$,

$$d(x) = \ln x - \sqrt{x} \leq d(4) = 2\ln 2 - 2 < 0 \text{ because } \ln 2 < 1.$$

  Thus for any real $x \geq 4$, $0 \leq \ln x \leq \sqrt{x}$ and when dividing by x, we obtain

$$0 \leq \frac{\ln x}{x} \leq \frac{\sqrt{x}}{x} = \frac{1}{\sqrt{x}}.$$

  The squeeze theorem allows us to conclude that $\lim_{x \to +\infty} \dfrac{\ln x}{x} = 0^+$.

- Setting for any $\alpha > 0$, $t = x^\alpha$ which tends to $+\infty$ when x tends to $+\infty$, we obtain

$$\frac{\ln x}{x^\alpha} = \frac{\ln t^{\frac{1}{\alpha}}}{t} = \frac{\ln t}{\alpha t}$$

which tends to 0^+ at $+\infty$ according to the previous result.

- We apply the previous result to $\dfrac{(\ln x)^\beta}{x^\alpha} = \left(\dfrac{\ln x}{x^{\frac{\alpha}{\beta}}}\right)^\beta$ with $\frac{\alpha}{\beta} > 0$ and obtain *3*.

- By setting $y = e^x$ we have that $x = \ln y$ and y tends to $+\infty$ when x tends to $+\infty$ so by writing

$$\frac{e^{\beta x}}{x^\alpha} = \frac{y^\beta}{(\ln y)^\alpha}$$

and applying the previous result we obtain *4* and *5*.

- We set $t = \dfrac{1}{x}$ which tends to $+\infty$ when x tends to 0^+ we have that

$$x^\alpha \ln x = \left(\frac{1}{t}\right)^\alpha \ln \frac{1}{t} = -\frac{\ln t}{t^\alpha}$$

which, according to *2*, tends to 0^- when t goes to $+\infty$, thus proving *6* and *7*.

3.2.2 Trigonometric functions

Here we summarise the classical properties for the cosine and sine functions after giving their definition through the trigonometric circle. The first thing we obtain from the trigonometric circle is the fundamental relation

$$\cos^2(x) + \sin^2(x) = 1, \quad \forall x \in \mathbb{R}.$$

A rigorous definition for the sine and cosine functions would call for the introduction of the exponential of a complex number. Here, we accept that such an object exists and we define the cos et sin functions using the relation

$$\exp(a + ib) = \exp(a)(\cos(b) + i \sin(b)), \quad \forall a, b \in \mathbb{R}^2.$$

In particular, we have the Euler formulae

$$\cos(b) = \frac{\exp(ib) + \exp(-ib)}{2}, \quad \sin(b) = \frac{\exp(ib) - \exp(-ib)}{2i}.$$

This properties are very important as they allow us to obtain all of the classical trigonometric formulae (addition, multiplication, duplication) from the equation

$$\exp(i(a + b)) = \exp(ia)\exp(ib).$$

Proposition 3.17. *The cosine function verifies the following properties*

1. *The* cos *function is defined, continuous and differentiable (and even belongs to the* $\mathcal{C}^\infty$ *space) on* $\mathbb{R}$ *and* $\cos'(x) = -\sin(x)$
2. *The* cos *function is* 2π*-periodic (*$\cos(x+2\pi) = \cos(x)$, $\forall x \in \mathbb{R}$*) and even.*
3. $\cos(x+\pi) = -\cos(x), \quad \cos\left(\frac{\pi}{2} - x\right) = \sin(x), \ \forall x \in \mathbb{R}$
4. $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b), \quad \forall a, b \in \mathbb{R}$
5. $\cos(2x) = \cos^2(x) - \sin^2(x), \ \forall x \in \mathbb{R}$

Proposition 3.18. *The sine function verifies the following properties*

1. *The* sin *function is defined, continuous and differentiable (and even belongs to the* $\mathcal{C}^\infty$ *space) on* $\mathbb{R}$ *and* $\sin'(x) = \cos(x)$
2. *The* sin *function is* 2π*-periodic (*$\sin(x+2\pi) = \sin(x)$, $\forall x \in \mathbb{R}$*) and odd.*
3. $\sin(\pi+x) = -\sin(x), \quad \sin\left(\frac{\pi}{2} - x\right) = \cos(x), \ \forall x \in \mathbb{R}$
4. $\sin(a+b) = \sin(a)\cos(b) + \sin(b)\cos(a), \quad \forall a, b \in \mathbb{R}$
5. $\sin(2x) = 2\sin(x)\cos(x), \ \forall x \in \mathbb{R}$

We immediately note that sin and cos verify the differential equation $u''(x) + u(x) = 0$. In a similar way to the exponential function, we could define the cos function as the unique solution to

$$u''(x) + u(x) = 0, \quad u(0) = 1, \quad u'(0) = 0.$$

To define the sin function, we can simply change the initial conditions: $u(0) = 0, u'(0) = 1$. Let us now introduce the tangent function

Definition 3.19. *Let* $x \in \mathbb{R}$*: when it is well defined, we call* $\tan(x)$ *the quotient of* $\sin(x)$ *and* $\cos(x)$*.*

Proposition 3.20. *The tangent function has the following properties:*

1. *The* tan *function is defined on* $\mathcal{D}_{\tan} = \bigcup_{k \in \mathbb{Z}} \left]-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right[$.
2. *The* tan *function is* π*-periodic and odd.*
3. *The* tan *function is differentiable on* $\mathcal{D}_{\tan}$ *and* $\tan'(x) = 1 + \tan^2(x)$, $\forall x \in \mathcal{D}_{\tan}$.
4. $\lim\limits_{x \to \frac{\pi}{2}^-} \tan(x) = +\infty, \qquad \lim\limits_{x \to -\frac{\pi}{2}^+} \tan(x) = -\infty$
5. $\lim\limits_{x \to 0} \frac{\tan(x)}{x} = \tan'(0) = 1$

6. $\tan(a+b) = \dfrac{\tan(a)+\tan(b)}{1-\tan(a)\tan(b)}$ *for any* $a, b \in \mathcal{D}_{\tan}$ *such that* $a+b \in \mathcal{D}_{\tan}$.

7. $\cos(2x) = \dfrac{1-\tan^2(x)}{1+\tan^2(x)}, \quad \sin(2x) = \dfrac{2\tan(x)}{1+\tan^2(x)}, \ \forall x \in \mathcal{D}_{\tan}$

The last formula is particularly useful to compute integrals that involve sin and cosine functions. Note once again that we have an alternative definition for the tan function on $]-\pi/2, \pi/2[$ using a differential equation: tan is solution to

$$u'(x) = 1 + u^2(x), \quad \forall x \in]-\pi/2, \pi/2[, \quad u(0) = 0.$$

3.2.3 Hyperbolic functions

We define the hyperbolic sin and cosine functions, denoted ch, sh as

$$\mathrm{ch}(x) = \frac{\exp(x)+\exp(-x)}{2}, \quad \mathrm{sh}(x) = \frac{\exp(x)-\exp(-x)}{2}, \ \forall x \in \mathbb{R}$$

Proposition 3.21.

1. *The sh and ch functions are differentiable on* $\mathbb{R}$ *with* $ch'(x) = sh(x)$ *and* $sh'(x) = ch(x)$.

2. *The ch function is even, the sh function is odd.*

3. *The sh function is increasing on* $\mathbb{R}$, $\lim_{x\to+\infty} sh(x) = +\infty$ *and* $\lim_{x\to-\infty} sh(x) = -\infty$.

4. *The ch function is non increasing on* $\mathbb{R}^-$ *and non decreasing on* $\mathbb{R}^+$ *and* $\lim_{x\to\pm\infty} ch(x) = +\infty$.

5. $\lim_{x\to 0} \dfrac{ch(x)-1}{x^2} = \dfrac{1}{2}, \quad \lim_{x\to 0} \dfrac{sh(x)}{x} = 1$

The sh function is a bijection from $\mathbb{R}$ to $\mathbb{R}$: we denote as $\mathrm{Argsh} = \mathrm{sh}^{-1}$ its inverse function. The ch function is a bijection from $\mathbb{R}^+$ to $[1, +\infty[$: we denote as $\mathrm{Argch} = \mathrm{ch}^{-1}$ its inverse function. We will study the behaviour of these inverse functions at the end of this course. The following formulae are reminiscent of the trigonometric equalities.

Proposition 3.22.

1. $ch(a+b) = ch(a)ch(b) + sh(a)sh(b), \quad \forall a, b \in \mathbb{R}$

2. $sh(a+b) = sh(a)ch(b) + sh(b)ch(a), \quad \forall a, b \in \mathbb{R}.$

3. $ch^2(x) - sh^2(x) = 1, \quad \forall x \in \mathbb{R}$

We note that sh and ch both verify the differential equation

$$u''(x) - u(x) = 0, \quad \forall x \in \mathbb{R}. \quad (E)$$

This gives another definition for the sh and ch functions: ch is the solution to (E) such that $u(0) = 1$ and $u'(0) = 0$ whereas sh is solution to (E) such that $u(0) = 0$ and $u'(0) = 1$.

We will study this more in depth during the tutorial classes (see the exercises from Section 6.2 of Chapter 6 of the tutorial exercises).

3.3 Some classical inequalities

3.3.1 Absolute value function and triangular inequalities

Definition 3.23. *We define the absolute value function of $x \in \mathbb{R}$, denoted as $|x|$, by*

$$|x| = x, \quad \forall x \geq 0 \text{ and,} \quad |x| = -x, \quad \forall x \leq 0$$

The absolute value function is defined on $\mathbb{R}$, even, continuous on $\mathbb{R}$, differentiable on $\mathbb{R}^*$ and non differentiable at 0. This function is of paramount importance in analysis, in particular because of the triangular inequalities:

Proposition 3.24. *1. First triangular inequality:*

$$\boxed{|x+y| \leq |x| + |y|, \quad \forall x, y \in \mathbb{R}}$$

2. Second triangular inequality:

$$\boxed{||x| - |y|| \leq |x+y|, \quad \forall x, y \in \mathbb{R}}$$

The use of inequalities that involve the absolute value function can be difficult to handle and we often prefer to use inequalities without absolute values:

Proposition 3.25. *For any $x \in \mathbb{R}$ et $M \geq 0$, we have*

$$|x| \leq M \quad \Leftrightarrow -M \leq x \leq M \quad \Leftrightarrow x \leq M \textbf{ and } -x \leq M.$$

The name triangular inequality comes from the geometrical interpretation of Proposition 3.24 when the numbers under consideration are complex numbers and the absolute value is replaced by the modulus.

Proposition 3.26. *For any $z_1, z_2 \in \mathbb{C}$, we have*

$$||z_1| - |z_2|| \leq |z_1 + z_2| \leq |z_1| + |z_2|.$$

Proof We first prove the first triangular inequality. Let $z_1, z_2 \in \mathbb{C}$ then:

$$|z_1 + z_2|^2 = (z_1 + z_2)(\bar{z}_1 + \bar{z}_2) = |z_1|^2 + 2Re(z_1\bar{z}_2) + |z_2|^2.$$

But we know that $|Re(z_1\bar{z}_2)| \leq |z_1\bar{z}_2| = |z_1||z_2|$. We deduce that

$$|z_1 + z_2|^2 \leq |z_1|^2 + 2|z_1||z_2| + |z_2|^2 = (|z_1| + |z_2|)^2.$$

By applying the square root, the property is proven. We now prove the second inequality in the following way:

$$|z_1| = |z_1 + z_2 - z_2| \leq |z_1 + z_2| + |z_2|.$$

Thus for any z_1, z_2, we have that

$$|z_1| - |z_2| \leq |z_1 + z_2| \qquad \text{and} \qquad |z_2| - |z_1| \leq |z_1 + z_2|.$$

The second triangular inequality is thus proven.
The geometrical meaning of the first triangular inequality is quite clear: the length of a side of a triangle is always smaller than the sum of the lengths of the two opposite sides. In a nutshell, the straight line is the shortest path.

3.3.2 Cauchy Schwarz inequality, Young inequality

We consider the inequality $|Re(z_1\bar{z}_2)| \leq |z_1||z_2|$ and we set $z_j = x_j + iy_j$ for j=1,2. We obtain an inequality known as the Cauchy Schwarz inequality:

$$|x_1x_2 + y_1y_2| \leq \sqrt{x_1^2 + y_1^2}\sqrt{x_2^2 + y_2^2}$$

The result extends to any pair of vectors $\mathbf{x} = (x_1, x_2, \cdots, x_d)$ and $\mathbf{y} = (y_1, y_2, \cdots, y_d)$ belonging to $\mathbb{R}^d$.

Proposition 3.27. *For any $(x_1, x_2, \cdots, x_d), (y_1, y_2, \cdots, y_d) \in \mathbb{R}^d$, we have*

$$\boxed{\left|\sum_{j=1}^{d} x_j y_j\right|^2 \leq \left(\sum_{j=1}^{d} x_j^2\right)\left(\sum_{j=1}^{d} y_j^2\right)}$$

The proof is based on an elementary property of second degree real polynomials.
Proof We define a second degree polynomial $P : \mathbb{R} \to \mathbb{R}$ as

$$P(t) = \sum_{j=1}^{d}(x_j + t\,y_j)^2 = \left(\sum_{j=1}^{d} y_j^2\right)t^2 + 2t\left(\sum_{j=1}^{d} x_j y_j\right) + \sum_{j=1}^{d} x_j^2 \geq 0, \quad t \in \mathbb{R}.$$

The degree of polynomial P is 2 and its sign is constant so its discriminant is non positive. The inequality thus obtained is precisely the Cauchy Schwarz inequality.

Another useful inequality is the Young inequality. We start by giving a simplified version:

Proposition 3.28. *For any $a, b \in \mathbb{R}$, we have*

$$\boxed{ab \leq \frac{a^2}{2} + \frac{b^2}{2}.}$$

The proof is based on the inequality $(a - b)^2 \geq 0$. For any $C > 0$, we have a more general version

$$ab \leq C\frac{a^2}{2} + \frac{1}{C}\frac{b^2}{2}.$$

We note that this inequality allows us to give an alternative proof of the Cauchy Schwarz inequality: for any $C > 0$, we have

$$\sum_{j=1}^{d} |x_j||y_j| \leq C\frac{\sum_{j=1}^{d} x_j^2}{2} + \frac{1}{C}\frac{\sum_{j=1}^{d} y_j^2}{2}.$$

By choosing $C = \sqrt{\dfrac{\sum_{j=1}^{d} y_j^2}{\sum_{j=1}^{d} x_j^2}}$, we obtain the Cauchy Schwarz inequality.

3.3.3 Inequality and convexity

Definition 3.29.

1. *A function* $f : I \subset \mathbb{R} \to \mathbb{R}$ *(with* I *an interval) is* ***convex*** *if and only if*

$$\boxed{\forall x, y \in I, \quad f(tx + (1-t)y) \leq t\, f(x) + (1-t)\, f(y), \quad \forall t \in [0,1].}$$

2. *A function* $f : I \subset \mathbb{R} \to \mathbb{R}$ *(with* I *an interval) is* ***concave*** *if and only if*

$$\boxed{\forall x, y \in I, \quad f(tx + (1-t)y) \geq t\, f(x) + (1-t)\, f(y), \quad \forall t \in [0,1].}$$

A function's convexity is hard to prove directly, we often use the following characterisations.

Proposition 3.30.

1. *Let* $f : I \to \mathbb{R}$ *a differentiable function:* f *is convex if and only if* f' *is an non decreasing function.*

2. *Let* $f : I \to \mathbb{R}$ *a differentiable function:* f *is convex if and only if* $f(y) \geq f(x) + f'(x)(y - x)$ *for any* $x, y \in I$*.*

3. *Let* $f : I \to \mathbb{R}$ *a twice differentiable function:* f *is convex if and only if* $f''(x) \geq 0$ *for any* $x \in I$*.*

The second characterisation has a geometrical meaning: the function f is convex if and only if its curve is always above its tangents. Note that the exp function is convex and the ln function is concave. The function $x \mapsto 1/x$ is convex on $\mathbb{R}^+$. Using the convexity of exp, we prove the Young inequality:

Proposition 3.31. *Let* $p, q > 0$ *such that* $\dfrac{1}{p} + \dfrac{1}{q} = 1$ *then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}, \quad \forall a > 0, \forall b > 0.$$

Proof We have

$$ab = \exp(\ln(a) + \ln(b)) = \exp\left(\frac{\ln a^p}{p} + \frac{\ln b^q}{q}\right) \leq \frac{1}{p}\exp(\ln a^p) + \frac{1}{q}\exp(\ln(b^q)) = \frac{a^p}{p} + \frac{b^q}{q}.$$

The inequality is then obtain through the convexity of exp.

Using the fact that the ln function is convex, we can prove the arithmetic-geometric inequality.

Proposition 3.32. *For any* $x_1 > 0, \cdots, x_d > 0$*, we have*

$$(x_1 x_2 \dots x_d)^{\frac{1}{d}} \leq \frac{1}{d}\sum_{i=1}^{d} x_i.$$

Proof The ln function is concave so we have

$$\ln\left(\frac{1}{d}\sum_{j=1}^{d} x_j\right) \geq \frac{1}{d}\sum_{j=1}^{d} \ln(x_j).$$

Applying the exponential function, we obtain the inequality that we want.

Using the fact that the sin function is concave on $\left[0, \frac{\pi}{2}\right]$, we obtain a *lower bound* for sin:

$$\sin(x) \geq \frac{2}{\pi}x, \quad \forall x \in \left[0, \frac{\pi}{2}\right].$$

Chapter 4

Fundamental theorems of analysis

4.1 Further information on limits and neighbourhoods

Throughout this chapter, $\mathcal{D}_f$ is the definition domain of function f.

4.1.1 A brief introduction to the topology of the real line

Topology is a structure that is imposed upon sets to allow us to discuss the continuity of any application T from a set E to a set F. Fully introducing the topology of $\mathbb{R}$ exceeds the goals of this course. We limit ourselves to two notions of topology that will be useful in the following: the neighbourhood of a point and the closure of a subset X of $\mathbb{R}$.

Notion of neighbourhood

The notion of neighbourhood was first mentioned in Chapter 2 as well as the extended real line. Recall that

$$\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty, -\infty\}.$$

We recall that $]a-\eta, a+\eta[$ is a neighbourhood of $a \in \mathbb{R}$ for any $\eta > 0$, and that $]A, +\infty[$ and $]-\infty, A[$ are neighbourhoods of $+\infty$ and of $-\infty$ for any $A \in \mathbb{R}$.

As well as giving more subtle characterisations of the convergence of a sequence of real numbers and the existence of a limit, the following definitions show how the extended real line and neighbourhoods can simplify some statements.

Definition 4.1. *The sequence of real numbers $(x_n)_{n\in\mathbb{N}}$ converges to $a \in \overline{\mathbb{R}}$ if and only if*

$$\forall V \in \mathcal{V}(a),\ \exists n_0 \in \mathbb{N},\ \forall n \in \mathbb{N}, \quad n \geq n_0 \Longrightarrow x_n \in V.$$

Definition 4.2. *A function f, defined on a neighbourhood of $a \in \overline{\mathbb{R}}$, tends to $\ell \in \mathbb{R}$ when x tends to a if and only if*

$$\forall \varepsilon > 0,\ \exists V \in \mathcal{V}(a),\ \forall x \in \mathcal{D}_f, \quad x \in V \implies |f(x) - \ell| < \varepsilon.$$

The following proposition is often useful in practice.

When a function verifies a property on a neighbourhood of a point

We say that a function f **verifies a property (P) on a neighbourhood of a point** $a \in \overline{\mathbb{R}}$ if there exists $V \in \mathcal{V}(a)$ such that on the set $V \cap \mathcal{D}_f$, f verifies (P). Note that $V \cap \mathcal{D}_f$ contains one of the following sets : $]a-h, a+h[$, $]a-h, a[$, $]a-h, a[\cup]a, a+h[$, $]a, a+h[$ if a is finite, the interval $]A, +\infty[$ if $a = +\infty$ and the interval $]-\infty, A[$ when $a = -\infty$. The next example illustrates such a statement.

Example 11. *f is* ***bounded on a neighbourhood*** *of $a \in \overline{\mathbb{R}}$ if there exists a neighbourhood V of a on which f is bounded,* i.e. *if there exists $M \in \mathbb{R}$ such that*

$$\forall x \in \mathcal{D}_f \cap V, \quad |f(x)| \leq M.$$

Proposition 4.3. *If a function f is defined on a neighbourhood $a \in \overline{\mathbb{R}}$ and tends to $\ell \in \mathbb{R}$ when x tends to a, then f is bounded on a neighbourhood of a.*

Proof When setting $\varepsilon = 1$ in the definition of a limit we obtain

$$\begin{aligned}
&\exists V \in \mathcal{V}(a),\ \forall x \in \mathcal{D}_f,\ x \in V &&\implies \quad |f(x) - \ell| < 1,\\
\iff &\exists V \in \mathcal{V}(a),\ \forall x \in \mathcal{D}_f,\ x \in V &&\implies \quad -1 < f(x) - \ell < 1,\\
\iff &\exists V \in \mathcal{V}(a),\ \forall x \in \mathcal{D}_f,\ x \in V &&\implies \quad \ell - 1 < f(x) < 1 + \ell,\\
\implies &\exists V \in \mathcal{V}(a),\ \forall x \in \mathcal{D}_f,\ x \in V &&\implies \quad |f(x)| < \max(|\ell - 1|, |1 + \ell|).
\end{aligned}$$

This proves that f is bounded on V. □

Cauchy criteria

The following criteria, that we will accept as true is very useful. It allows us to know whether a limit exists without having to determine it.

Theorem 4.4. *Let f be a function defined on a neighbourhood of a point $a \in \mathbb{R}$. Then f has a finite limit $\ell \in \mathbb{R}$ when x tends to a if and only if it verifies the following property called* ***Cauchy criteria***

$$\forall \varepsilon > 0,\ \exists V \in \mathcal{V}(a),\ \forall (x, x') \in V^2, |f(x) - f(x')| \leq \varepsilon.$$

4.1.2 Bounding criteria

We often determine whether a function converges at a point by both upper bounding and lower bounding it with two more simple functions that converge to the same limit. We first have the following lemma that is often used to show that a function has a limit at a point $a \in \overline{\mathbb{R}}$.

Proposition 4.5. *Let a function f be defined on a neighbourhood of a point $a \in \overline{\mathbb{R}}$. We suppose that there exists $\ell \in \mathbb{R}$, $W \in \mathcal{V}(a)$ and a function h defined on $W \cap \mathcal{D}_f$ such that*

$$\lim_{x \to a} h(x) = 0 \text{ and } \quad \forall x \in W \cap \mathcal{D}_f,\ |f(x) - \ell| \leq h(x).$$

We then have

$$\lim_{x \to a} f(x) = \ell.$$

Proof Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of points belonging to $\mathcal{D}_f$ such that $\lim_{n\to+\infty} x_n = a$. As W is a neighbourhood of a, there exists n_0 such that $x_n \in W$ for any $n \geq n_0$. We can then write

$$|f(x_n) - \ell| \leq h(x_n) \text{ for any } n \geq n_0.$$

As $\lim_{x\to 0} h(x) = 0$, we deduce that $\lim_{n\to+\infty} h(x_n) = 0$. We then directly obtain the desired conclusion from the definition of the convergence of the sequence $(f(x_n))_{n\in\mathbb{N}}$ to ℓ. $\square$

Example 12. *Determine, if it exists, the limit when x tends to 0 of the function defined by*

$$\frac{\sin(x^2\ln(x))}{x}.$$

Answer : This function is defined for $x > 0$. We have

$$\left|\frac{\sin(x^2\ln(x))}{x}\right| = \left|\frac{\sin(x^2\ln(x))}{x^2\ln(x)}\right| |x\ln(x)| \leq |x\ln(x)|\,.$$

As $\lim_{x\to 0} x\ln(x) = 0$, we deduce from the previous proposition that

$$\lim_{x\to 0} \sin(x^2\ln(x))/x = 0.$$

Proposition 4.6. *Let f, g and h be three functions defined on a subset D of $\mathbb{R}$ that verify $f \leq g \leq h$ on a neighbourhood of $a \in \mathbb{R}$. If $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = \ell \in \mathbb{R}$, then*

$$\lim_{x\to a} g(x) = \ell.$$

Proof Left to the reader as an exercise. $\square$

The following result is often useful in practice.

Theorem 4.7. *Let f be a function that tends to 0 when x tends to $a \in \overline{\mathbb{R}}$ and let g a bounded function on a neighbourhood of a. We then have*

$$\lim_{x\to a}(gf)(x) = 0$$

Proof If $(u_n)_{n\in\mathbb{N}} \subset \mathcal{D}_{gf}$ is such that $\lim_{n\to+\infty} u_n = a$, then $(gf)(u_n) = g(u_n)f(u_n)$ is the product of a bounded sequence $(g(u_n))_{n\in\mathbb{N}}$ and a sequence $(f(u_n))_{n\in\mathbb{N}}$ that tends to 0. According to Theorem 2.12 it holds that $\lim_{n\to+\infty}(gf)(u_n) = 0$. $\square$

Exercise 7. *Show that*

$$\lim_{x\to 0} x\sin(1/x) = 0.$$

We can also show that a function diverges towards $\pm\infty$ when x tends to $a \in \overline{\mathbb{R}}$ by using the following theorem.

Theorem 4.8. *Let f and g be two functions defined on a neighbourhood V of $a \in \overline{\mathbb{R}}$ such that*

$$\forall x \in V,\ f(x) \leq g(x). \tag{4.1}$$

We then have

1. $\lim_{x\to a} f(x) = +\infty \Rightarrow \lim_{x\to a} g(x) = +\infty$,
2. $\lim_{x\to a} g(x) = -\infty \Rightarrow \lim_{x\to a} f(x) = -\infty$.

Proof

1. If $\lim_{x\to a} f(x) = +\infty$ then

   $$\forall A > 0,\ \exists W \in \mathcal{V}(a),\ \forall x \in \mathcal{D}_f, \quad x \in W \Longrightarrow f(x) > A$$

   and according to assumption (4.1) we have

   $$\forall A > 0,\ \exists W \in \mathcal{V}(a),\ \forall x \in \mathcal{D}_g, \quad x \in V \cap W \Longrightarrow g(x) > A$$

   so $\lim_{x\to a} g(x) = +\infty$.

2. Similarly, if $\lim_{x\to a} g(x) = -\infty$ then

   $$\forall B < 0,\ \exists W \in \mathcal{V}(a),\ \forall x \in \mathcal{D}_g, \quad x \in W \Longrightarrow g(x) < B$$

   and according to assumption (4.1) we have

   $$\forall B < 0,\ \exists W \in \mathcal{V}(a),\ \forall x \in \mathcal{D}_f, \quad x \in V \cap W \Longrightarrow f(x) < B$$

   so $\lim_{x\to a} f(x) = -\infty$.

□

4.1.3 Limit and sign

There is a close link between the sign of a function f on a neighbourhood of a point $a \in \overline{\mathbb{R}}$ and the sign of $\lim_{x\to a} f(x)$, when the limit exists, as shown by the following theorems whose results are fundamental in analysis.

Theorem 4.9. *Let f by a function whose limit at $a \in \overline{\mathbb{R}}$ is ℓ. If $\ell > 0$ (resp. $\ell < 0$), there exists a neigbourhood V of a such that*

$$\forall x \in V \cap \mathcal{D}_f, f(x) > 0 \text{ (resp. } f(x) < 0\text{)}.$$

Proof We only need to prove the proposition for $\ell > 0$, as the other case where $\ell < 0$ can be deduced by taking the function $-f$. We set $\varepsilon = \ell/2 > 0$. As $\lim_{x \to a} f(x) = \ell$, there exists a neighbourhood V of a such that

$$\forall x \in V \cap \mathcal{D}_f,\ |f(x) - \ell| \leq \ell/2, \quad i.e \quad f(x) - \ell/2 \leq f(x) - \ell \leq \ell/2.$$

It then follows that for any $x \in V \cap \mathcal{D}_f$, $f(x) \geq \ell - \ell/2 = \ell/2 > 0$. $\square$

Remark 6. *The fact that ℓ is > 0 or < 0 is key. Indeed the function $x \mapsto x\sin(1/x)$ tends to 0 when x tends to 0 as the product of a function that is bounded on a neighbourhood of 0 : $x \mapsto \sin(1/x)$, and a function $x \mapsto x$, tending to 0 when x tends to 0. It changes sign an infinite number of times on any interval $]-h, 0[$ or $]0, h[$ for any $h > 0$.*

We also have the following theorem that is sometimes referred to as to "*principle of inequality conservation at the limit* ".

Theorem 4.10. *Let f be a function and V a neighbourhood of $a \in \overline{\mathbb{R}}$ such that*

$$\forall x \in V \cap \mathcal{D}_f, \quad f(x) \geq 0 \ \textit{(resp. } f(x) \leq 0\textit{)}.$$

Then, if f has a limit at a,

$$\lim_{x \to a} f(x) \geq 0 \ \textit{(resp. } \leq 0\textit{)}.$$

Proof Suppose that $\lim_{x \to a} f(x) < 0$ (resp. $\lim_{x \to a} f(x) > 0$). The previous theorem shows that there exists a neighbourhood V' of a such that

$$\forall x \in V' \cap \mathcal{D}_f, \quad f(x) < 0 \text{ (resp. } f(x) > 0\text{)}.$$

This contradicts the fact that $\forall x \in V \cap \mathcal{D}_f$, $f(x) \geq 0$ (resp. $f(x) \leq 0$), and is thus absurd. $\square$

Remark 7. *The previous theorem can be written using mathematical symbols*

$$f(x) \geq 0 \implies \lim_{x \to a} f(x) \geq 0.$$

*One does however have to be careful about the fact that $f(x) > 0$ only implies that $\lim_{x \to a} f(x) \geq 0$. For example, the function $x \mapsto e^{-x}$ is > 0 for any $x \in \mathbb{R}$. But $\lim_{x \to -\infty} e^{-x} = 0$. We thus note for this property that "**strict inequalities become weak at the limit**".*

4.2 Fundamental theorems for continuous functions

4.2.1 Extensions on continuity

Throughout this section, I is a non empty interval containing more than one point.

Jump discontinuity

Definition 4.11. *Let a function f be defined on I and let $a \in I$. The function f has a* jump discontinuity at *a if both its left-hand and right-hand limits are finite and*

$$\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x) \, .$$

Exercise 8. *Amongst the non continuous functions in exercise 6, which ones have jump discontinuities ?*

We note that the functions that have a jump discontinuity at a are sometimes not defined at a.

We previously saw that monotonous functions always have finite left-hand and right-hand limits at any point of their definition domain. They thus can only have jump discontinuities. We can show (doing so is slightly technical) the following property.

Proposition 4.12. *Any monotonous function $f : I \to \mathbb{R}$ (with I an interval) has a finite or countable number of discontinuities.*

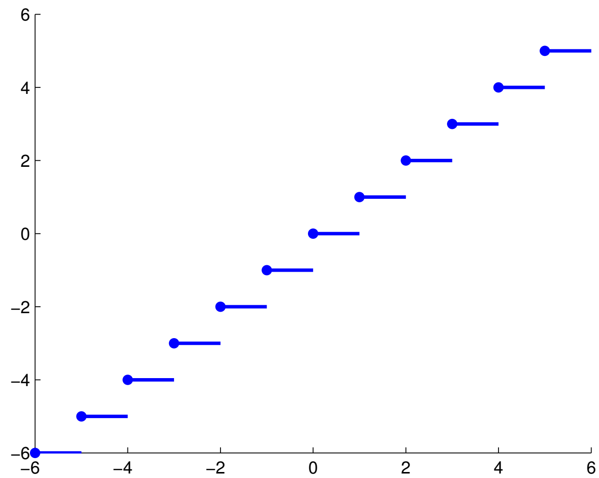
Piecewise continuity

Definition 4.13. *A function f is said to be* ***piecewise continuous*** *on I if f is continuous on any segment $[a, b] \subset I$ where $a < b$ except on* a finite number of points *of $[a, b]$ where it has jump discontinuities.*

Example 13. *The* floor *function returns for any given $x \in \mathbb{R}$ the biggest of the integers in $n \in \mathbb{Z}$ that are $\leq x$. It is denoted as*

$$\lfloor x \rfloor = \max \{ n \in \mathbb{Z} \mid n \leq x \} \, .$$

The following figure shows the graph of the floor *function on the interval* $[-6, 6]$



The dots denote the values of the function at the points of discontinuity. It clearly is a piecewise continuous function on $\mathbb{R}$. *Its restriction to the interval* $[0, +\infty[$ *is often called whole part function and denoted* $\mathrm{E}(x)$.

Continuous extension

We saw previously that the function $f : x \mapsto \sin(x)/x$ is not defined at 0 but that $\lim_{x\to 0} f(x) = 1$. By defining the function $\tilde{f}$ as $\tilde{f}(x) = \sin(x)/x$ if $x \neq 0$ and $\tilde{f}(0) = 1$, we obtain a function that is continuous at 0.
This method can be applied to any function that is continuous on an interval I, not defined at a real number a but that has a finite limit at a: the function that is then obtain is called the *continuous extension.*

Definition 4.14. *Let* f *be a function that is defined and continuous on* I *except at* $a \in I$, *such that* $\lim_{x\to a} f(x) = l$, *with* $l \in \mathbb{R}$.
We define on I *the function* $\tilde{f} : x \mapsto \begin{cases} f(x) & si\ x \neq a \\ l & si\ x = a \end{cases}$.

Then the function $\tilde{f}$ *is continuous on* I *and called the* continuous extension *of* f. *We can also say that* f *can be extended through continuity at* a.

Exercise 9. *We define two functions* f_1 *and* f_2 *for any non zero* x *as*

$$f_1(x) = x^2 \ln x^2 \quad and \quad f_2(x) = x/|x|.$$

Can these functions be extended through continuity at 0*?*

4.2.2 Weierstrass theorem

Let f be a function defined on an interval I. In general, if we do not know whether f is continuous on I then we cannot say anything on the existence of a solution for the following optimisation problem:

$$\begin{cases} m \in I, \\ f(m) \leq f(x), \ \forall x \in I. \end{cases}$$

This problem comes up frequently in engineering and in most fields that call for mathematical modelling. Generally, f is a criterion: cost, safety coefficient, distance to a goal etc. that we want to minimise.
If I is a segment $[a, b]$ and if $f \in \mathcal{C}^0([a, b])$, then we know that there exists a solution. Before studying this problem, we first introduce some definitions. If I is a non empty part of $\mathcal{D}_f$, $\inf f(I)$ is usually denoted

$$\inf f(I) := \inf_{x \in I} f(x).$$

If $\inf f(I)$ is attained ($\inf f(I) = f(x_0)$ with $x_0 \in I$), we then have a **minimum** for f on I and we denote it

$$\inf f(I) = \min_{x \in I} f(x).$$

We also have

$$\sup f(I) := \sup_{x \in I} f(x)$$

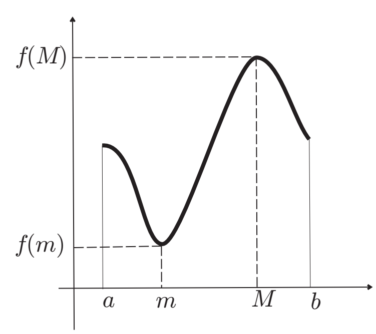
and we have a **maximum** for f on I if $\sup f(I)$ is attained and we denote it

$$\sup f(I) = \max_{x \in I} f(x).$$

Theorem 4.15 (Weierstrass theorem)**.** *Let f be a continuous function on a segment $[a, b]$, then f is bounded and reaches its bounds.*

In other words, two real numbers m and M exist such that

1. $\forall x \in [a, b], m \leq f(x) \leq M$,
2. $\exists\ \alpha \in [a, b], f(\alpha) = m = \min_{x \in [a,b]} f(x)$ and $\exists\ \beta \in [a, b], f(\beta) = M = \max_{x \in [a,b]} f(x)$.



Proof

We first prove by contradiction that f is upper bounded. Let us suppose that f is not upper bounded, this means that

$$\forall A \in \mathbb{R}, \exists x \in [a, b], f(x) > A.$$

In particular

- for $A = 1, \exists x_1 \in [a, b], f(x_1) > 1$,
- for $A = 2, \exists x_2 \in [a, b], f(x_2) > 2$,
- for $A = 3, \exists x_3 \in [a, b], f(x_3) > 3$,

$\vdots$

- for $A = n \in \mathbb{N}^*, \exists x_n \in [a, b], f(x_n) > n$.

We thus construct a sequence $(x_n)_{n\in\mathbb{N}^*}$ such that

$$\forall n \in \mathbb{N}^*, x_n \in [a, b] \text{ and } \lim_{n\to+\infty} f(x_n) = +\infty. \tag{4.2}$$

According to the Bolzano-Weierstrass theorem (*cf.* Theorem **??**), as the sequence $(x_n)_{n\in\mathbb{N}^*}$ is real and bounded, there exists a subsequence $(u_n)_{n\in\mathbb{N}^*} = (x_{\varphi(n)})_{n\in\mathbb{N}^*}$ that converges to $c \in [a, b]$. As f is continuous on $[a, b]$, it then follows that the sequence $(f(u_n))_{n\in\mathbb{N}^*}$ converges to $f(c)$. But according to (4.2), as the sequence $(f(u_n))_{n\in\mathbb{N}^*}$ is a subsequence of $(f(x_n))_{n\in\mathbb{N}^*}$, it diverges towards $+\infty$. These two results are contradictory, so our assumption is absurd and the function f is upper bounded.

We show that the function f is lower bounded using the same technique.

Let us now show that the function f reaches its bounds.

As the set $B = \{f(x), x \in [a, b]\}$ is a non empty, upper bounded and lower bounded subset of $\mathbb{R}$, it has an upper and a lower bound. We denote

$$m = \inf_{x\in[a,b]} f(x) \text{ and } M = \sup_{x\in[a,b]} f(x).$$

It remains to show that there exists α and β in $[a, b]$ such that $f(\alpha) = m$ and $f(\beta) = M$. Through the definition of the upper bound,

$$\forall \varepsilon > 0,\ \exists y_\varepsilon \in [a, b],\ M - \varepsilon < f(y_\varepsilon) \leq M.$$

In particular,

$$\forall n \in \mathbb{N}^*, \text{ if } \varepsilon = \frac{1}{n},\ \exists y_n \in [a, b],\ M - \frac{1}{n} < f(y_n) \leq M.$$

We thus construct a sequence $(y_n)_{n\in\mathbb{N}^*}$ such that

$$\forall n \in \mathbb{N}^*, y_n \in [a, b] \text{ and } \lim_{n\to+\infty} f(y_n) = M. \tag{4.3}$$

Once again according to the Bolzano Weierstrass theorem, as the sequence $(y_n)_{n\in\mathbb{N}^*}$ is real and bounded, there exists a subsequence $(v_n)_{n\in\mathbb{N}^*} = (y_{\psi(n)})_{n\in\mathbb{N}^*}$ that converges to $\beta \in [a,b]$. As f is continuous on $[a,b]$, it follows that the sequence $(f(v_n))_{n\in\mathbb{N}^*}$ converges to $f(\beta)$. But according to (4.3), as the sequence $(f(v_n))_{n\in\mathbb{N}^*}$ is a subsequence of $(f(y_n))_{n\in\mathbb{N}^*}$, it converges to M. Through uniqueness of the limit, $f(\beta) = M$ so $M \in B$ and $M = \max\limits_{x\in[a,b]} f(x)$.

We show in the same way that there exists α in $[a,b]$ such that $f(\alpha) = m$ so $m \in B$ and $m = \min\limits_{x\in[a,b]} f(x)$. □

4.2.3 Intermediate value theorem

Similarly to above, for a function f defined on I, we consider the following equation

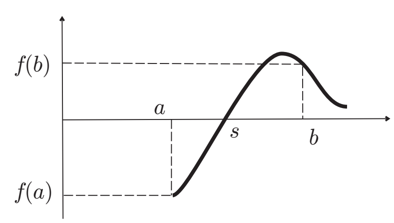
$$\begin{cases} s \in I, \\ f(s) = 0. \end{cases}$$

If we do not know whether the function f is continuous on I, then we generally cannot say anything on the existence of a solution. However, if we know that $f \in \mathcal{C}^0(I)$ and that f changes sign on I : that is that there exists $a < b$ in I such that

$$f(a)f(b) < 0$$

then we will be able to state that the previous problem has a solution s and that $s \in [a,b]$. This property is used in numerical analysis to approach the solution on computers using a dichotomy method. The following theorem, sometimes called the **Bolzano theorem** gives us this property.

Theorem 4.16 (Bolzano theorem)**.** *Let $a < b$ two real numbers. Let f be continuous on $[a,b]$ such that $f(a)f(b) < 0$. Then there exists $s \in]a,b[$ such that $f(s) = 0$.*



Proof We give the proof for the case where $f(a) < 0 < f(b)$. The other case where $f(a) > 0 > f(b)$ can be obtained by replacing f by $-f$.
Let

$$X = \{x \in [a,b] \mid f(x) < 0\}.$$

This set has the following properties :
- X is non empty : because $a \in X$,
- X is upper bounded : because b is an upper bound for X.
The supremum axiom then tells us that X has a supremum $s = \sup X$.

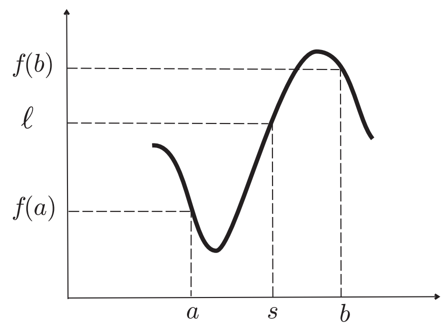


Figure 4.1: Illustration of the intermediate value theorem

Let us first note that s cannot be equal to a or to b because continuity of f tells us that the function is < 0 on a neighbourhood of a and > 0 on a neighbourhood of b. For the same reasons $s \notin X$, because otherwise X would have elements that are $> s$. We thus have $f(s) \geq 0$.
We now use the fact that the characterisation of supremums ensures, just like in the proof of the Weierstrass theorem, that there exists a sequence of points $(x_n)_{n\in\mathbb{N}}$ of X such that $\lim\limits_{n\to\infty} x_n = s$. Continuity of f then tells us that

$$\lim_{n\to\infty} f(x_n) = f(s) \leq 0 \text{ because } \forall n \in \mathbb{N}, x_n \in X \text{ so } f(x_n) < 0.$$

We thus have $f(s) \geq 0$ and $f(s) \leq 0$. It follows that $f(s) = 0$. $\square$

Remark 8.
The Bolzano theorem does not state that s is unique, but simply ensures the existence of s.

Example 19. *Let $f : I = [-2, 2] \to \mathbb{R}$ such that $f(x) = x^3 - x$ for any $x \in I$. We have $f(-2) = -6$ and $f(2) = 6$. The function f is continuous on I and 0 is between -6 and 6: there thus exists $c \in [-2, 2]$ such that $f(c) = 0$. We quickly see on this example that c is not unique: $c = -1, 0, 1$ all work.*

We deduce from the previous property the following fundamental theorem called the **intermediate value theorem.**

Theorem 4.17 (Intermediate value theorem)**.** *Let f be a continuous function on $[a, b]$. Define $m = \underset{x\in[a,b]}{inf} f(x)$ and $M = \underset{x\in[a,b]}{sup} f(x)$ then $f([a, b]) = [m, M]$.*
In other words, the image of an interval through a continuous function is an interval. In particular,

$$\forall \ell \in [m, M],\ \exists s \in [a, b],\ f(s) = \ell.$$

The name of this theorem comes from the fact that the real number ℓ is an intermediary value between $m = \inf\limits_{x\in[a,b]} f(x)$ and $M = \sup\limits_{x\in[a,b]} f(x)$ (see Figure 4.1).
Just like the Bolzano theorem, this theorem only ensures the existence of s, not its uniqueness. To ensure uniqueness, we need an extra assumption.

Theorem 4.18. *Let f a continuous and strictly monotonous function on $[a, b]$. Then f is a bijection from $[a, b]$ to $f([a, b])$.*

Proof As f is continuous on $[a, b]$, according to the intermediate value theorem $f([a, b])$ is an interval.
If f is increasing, $f([a, b]) = [f(a), f(b)]$ and if f is decreasing, we have $f([a, b]) = [f(b), f(a)]$.

Thus, according to the intermediate value theorem once again, for any real ℓ such that $f(a) \leq \ell \leq f(b)$ or $f(b) \leq \ell \leq f(a)$, there exists s such that $f(s) = \ell$.
Suppose that there exists $t \neq s$ such that $f(t) = \ell$. We then have $t < s$ or $s < t$. As f is strictly monotonous, we have $\ell = f(t) < f(s) = \ell$ or $\ell = f(s) < f(t) = \ell$ which is absurd. So for any real ℓ such that $f(a) \leq \ell \leq f(b)$ or $f(b) \leq \ell \leq f(a)$, there **exists** a **unique** $s \in [a, b]$ such that $f(s) = \ell$ and f is a bijection from $[a, b]$ to $f([a, b])$. $\square$

We can also show the following generalisation of this result.

Proposition 4.19. *Let $a < b$, a can be taken equal to $-\infty$ and b can be taken equal to $+\infty$. Let $f :]a, b[\to \mathbb{R}$ continuous and strictly monotonous then f is a bijection from $]a, b[$ to $]m, M[$ with $m = \min(\lim_{x \to a} f(x), \lim_{x \to b} f(x))$ and $M = \max(\lim_{x \to a} f(x), \lim_{x \to b} f(x))$.*

Use of the theorems to establish image sets and inverse images

When f is a continuous function on an interval I, $J = f(I)$ is also an interval of the same type. Let $(a, b) \in \mathbb{R}^2$.

1. When f is increasing we have the following results
   (a) if $I = [a, b]$ then $J = [f(a), f(b)]$
   (b) if $I =]a, b[$ then $J =]f(a), f(b)[$
   (c) if $I =]a, b]$ then $J =]f(a), f(b)]$
   (d) if $I = [a, b[$ then $J = [f(a), f(b)[$.
2. When f is decreasing we have the following results
   (a) if $I = [a, b]$ then $J = [f(b), f(a)]$
   (b) if $I =]a, b[$ then $J =]f(b), f(a)[$
   (c) if $I =]a, b]$ then $J = [f(b), f(a)[$
   (d) if $I = [a, b[$ then $J =]f(b), f(a)]$.

When $a = -\infty$ or $b = +\infty$, the same results hold when we replace $f(a)$ by $\lim_{x \to -\infty} f(x)$ and $f(b)$ by $\lim_{x \to +\infty} f(x)$.

Example 20. *Let us come back to the example of the function $f : x \mapsto x^3 - x$. From the bijection theorem and the variation table of f, we deduce:*

1. $f([0, +\infty[) = [-\frac{2}{3\sqrt{3}}; +\infty[$.

2. $f^{-1}([0, +\infty[) = [-1, 0] \cup [1, +\infty[$.

4.2.4 Lipschitz functions

Let us consider the first order polynomial function (we also say linear affine) $x \mapsto f(x) = ax+b$, with $a \neq 0$ and $b \in \mathbb{R}$. We already know that this function is defined and continuous on $\mathbb{R}$ by using operations on continuous functions. Let us however come back to the definition of continuity at $x \in \mathbb{R}$: for a given $\varepsilon > 0$ what is the number η that ensures that if $|x-y| \leq \eta$, then $|f(x) - f(y)| \leq \varepsilon$? We first compute $|f(x) - f(y)| = |ax + b - ay - b| = |a|\,|x-y|$. We can thus simply take $\eta = \varepsilon/\,|a|$ to ensure that $|f(x) - f(y)| \leq \varepsilon$ if $|x-y| \leq \eta$. The function f belongs to the class of functions that are called Lipschitz functions with Lipschitz parameter $|a|$ that we define as follows.

Definition 4.20. *Let $k \in \mathbb{R}_+^*$. A function f is **Lipschitz with Lipschitz constant** k (or k-**Lipschitz**) on the interval I if it is defined on I and verifies:*

$$\boxed{\forall (x,y) \in I^2, \quad |f(x) - f(y)| \leq k\,|x-y|\,.}$$

We immediately have the following property.

Proposition 4.21. *If f is k-Lipschitz on I, then it is continuous on I.*

Remark 9. *The converse is not true, as is shown by the following example. The function $x \mapsto \sqrt{x}$ is defined and continuous on the interval $[0, +\infty[$. If it were k-Lipschitz then by taking $x = 0$ and any $y \neq 0$, we would have*

$$|\sqrt{y} - \sqrt{0}| = \sqrt{y} \leq ky = k|y-0|,$$

which, when dividing each term by $\sqrt{y} > 0$ leads to

$$\frac{1}{\sqrt{y}} \leq k, \ \forall y > 0.$$

If we now take $y = 1/n^2$ with n an integer ≥ 1, this leads to

$$n \leq k, \ \forall n \textit{ integer } \geq 1,$$

which contradicts the fact that $\mathbb{N}$ is not bounded.

The following property states that any convex function is Lipschitz.

Proposition 4.22. *Let $a < b$ and $f :]a, b[\to \mathbb{R}$ a convex function then f is Lipschitz on any interval $[\alpha, \beta]$ where $a < \alpha < \beta < b$. In particular it is continuous on $[\alpha, \beta]$.*

Proof As the function f is convex then for any $\gamma \in]a, b[$, the function $r_\gamma : x \mapsto \dfrac{f(x) - f(\gamma)}{x - \gamma}$ is non decreasing. If we choose $\gamma \in]a, \alpha[$ et $\delta \in]\beta, b[$, we have for any $x, y \in]\alpha, \beta[$

$$r_\alpha(\gamma) \leq r_\alpha(x) = r_x(\alpha) \leq r_x(y) = r_y(x) \leq r_y(\beta) = r_\beta(y) \leq r_\beta(\delta).$$

We thus deduce that for any $x, y \in [\alpha, \beta]$, we have

$$r_\alpha(\gamma) \leq \frac{f(x) - f(y)}{x - y} \leq r_\beta(\delta).$$

Thus

$$|f(x) - f(x)| \leq k|x - y|,$$

with $k = \max\{|r_\alpha(\gamma)|, |r_\beta(\delta)|\}$. The function f is thus Lipschitz on $[\alpha, \beta]$. $\square$

4.2.5 Uniformly continuous functions

If a function f is k-Lipschitz, we saw above that, for any $x \in I$, the η that ensures that $|f(x) - f(y)| \leq \varepsilon$ if $|x - y| \leq \eta$ does not depend on the point x. The functions for which this property holds are said to be **uniformly continuous** on I.

Definition 4.23. *Let f defined on I. We say that f is* uniformly continuous *on I iff*

$$\boxed{\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall (x, y) \in I^2, \quad |x - y| \leq \eta \Longrightarrow |f(x) - f(y)| \leq \varepsilon.}$$

Remark 10. *The idea of uniform continuity is more restrictive that continuity. However, it is a global notion whereas continuity is a local notion: a function f is continuous on I if for any a in I, f is continuous at a i.e.*

$$\forall \varepsilon > 0,\ \exists \eta > 0,\ \forall x \in I, \quad |x - a| \leq \eta \Longrightarrow |f(x) - f(a)| \leq \varepsilon.$$

In the definition of continuity on I, the η depends on the point a and the chosen ε. In the definition of uniform continuity on I, the η only depends on the chosen ε.

We have already shown, when proving continuity of a k-Lipschitz function that a function that is k-Lipschitz on I is uniformly continuous on I. The two concepts are not equivalent though, as shown by the following example.

Example 14. *The function defined on $[0, +\infty[$ as $x \mapsto \sqrt{x}$, is uniformly continuous on this interval. This is a consequence of the following inequality that holds for any x and $y \geq 0$*

$$\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|}\ .$$

Indeed, the inequality is trivially verified if $x = y$. We can thus suppose that $x \neq y$ and consider

$$\frac{\left|\sqrt{x} - \sqrt{y}\right|}{\sqrt{|x - y|}} = \frac{\left|\sqrt{x} - \sqrt{y}\right|}{\sqrt{\left|\left(\sqrt{x}\right)^2 - \left(\sqrt{y}\right)^2\right|}} = \frac{\left|\sqrt{x} - \sqrt{y}\right|}{\sqrt{\left|\left(\sqrt{x} - \sqrt{y}\right)\left(\sqrt{x} + \sqrt{y}\right)\right|}}$$

$$= \frac{\left|\sqrt{x} - \sqrt{y}\right|}{\sqrt{\sqrt{x} + \sqrt{y}}\sqrt{\left|\sqrt{x} - \sqrt{y}\right|}} = \frac{\sqrt{\left|\sqrt{x} - \sqrt{y}\right|}}{\sqrt{\sqrt{x} + \sqrt{y}}} \leq 1.$$

So let $\varepsilon > 0$; for $\eta = \varepsilon^2$, if x, y are in $[0, +\infty[$ and verify $|x - y| \leq \eta$, it follows, according to the inequality above, that $\left|\sqrt{x} - \sqrt{y}\right| \leq \sqrt{|x - y|} \leq \sqrt{\eta} = \sqrt{\varepsilon^2} = \varepsilon$. This shows that the function $x \mapsto \sqrt{x}$ is uniformly continuous on $[0, +\infty[$.

Heine's theorem, that we now state, gives a sufficient condition to ensure that a function is uniformly continuous.

Theorem 4.24 (Heine's theorem)**.** *If a function f is continuous on a* segment $[a, b]$*, then f is uniformly continuous on* $[a, b]$.

Proof We prove the theorem by contradiction. We suppose that f is not uniformly continuous on $[a, b]$. So there exists $\varepsilon_0 > 0$ such that for any $\eta > 0$, we can find x_η and y_η that verify: $x_\eta,\ y_\eta \in [a, b]$, $|x_\eta - y_\eta| \leq \eta$ and $|f(x_\eta) - f(y_\eta)| \geq \varepsilon_0$. In particular, if, for any $n \in \mathbb{N}$, η is chosen of the form $\eta = 1/(n + 1)$, by calling x_n and y_n the corresponding x_η and y_η, we obtain

$$x_n,\ y_n \in [a, b]\,,\ |x_n - y_n| \leq 1/(n + 1) \text{ et } |f(x_n) - f(y_n)| \geq \varepsilon_0 \text{ for any } n \in \mathbb{N}.$$

The sequence $(x_n)_{n\in\mathbb{N}}$ belongs to $[a, b]$ and so is bounded. The Bolzano-Weierstrass theorem then ensures that there exists a subsequence, given by $u_n = x_{\varphi(n)}$ with φ increasing from $\mathbb{N}$ to $\mathbb{N}$, and $u \in \mathbb{R}$ such that $\lim\limits_{n\to+\infty} u_n = u$. As $a \leq u_n = x_{\varphi(n)} \leq b$, we have, when going to the limit, that $u \in [a, b]$. Let us consider the subsequence of $(y_n)_{n\in\mathbb{N}}$ related to the same indices as those of the subsequence of $(x_n)_{n\in\mathbb{N}}$, this is the sequence defined by $v_n = y_{\varphi(n)}$ pour tout $n \in \mathbb{N}$. We have

$$|u_n - v_n| = \left|x_{\varphi(n)} - y_{\varphi(n)}\right| \leq \frac{1}{(\varphi(n) + 1)} \leq \frac{1}{n + 1}$$

because any increasing application from $\mathbb{N}$ to $\mathbb{N}$ verifies $\varphi(n) \geq n$ for any n. We thus have

$$|v_n - u| = |v_n - u_n + u_n - u| \leq \frac{1}{n + 1} + |u_n - u|\,.$$

This shows that $\lim\limits_{n\to+\infty} v_n = u$. We thus obtain (through the continuity of f),

$$\begin{aligned}
\varepsilon_0 \leq \left|f(x_{\varphi(n)}) - f(y_{\varphi(n)})\right| &= |f(u_n) - f(v_n)| \\
&= |f(u_n) - f(u) + f(u) - f(v_n)| \\
&\leq |f(u_n) - f(u)| + |f(v_n) - f(u)|\,.
\end{aligned}$$

By going to the limit with the right-hand term of the inequality and using the fact that f is continuous at u, it follows that

$$\varepsilon_0 \leq 0 + 0 = 0.$$

This contradicts the hypothesis $\varepsilon_0 > 0$ and proves the theorem. □

Remark 11. *We cannot improve on the previous theorem. Let us illustrate this through two examples.*

1. *The function $x \mapsto 1/x$ is continuous on $]0, 1]$. We are going to prove ad absurdio that it is not uniformly continuous.*
   Suppose that $x \mapsto 1/x$ is uniformly continuous on $]0, 1]$. For any $\varepsilon > 0$, there exists

$\eta > 0$, such that $|1/x - 1/y| \leq \varepsilon$ as soon as $|x-y| \leq \eta$. Let $x \in \left]0, \frac{1}{2}\right]$. We set $\eta_2 = min\left(\eta, \frac{1}{2}\right)$ and $y = x + \eta_2$. We have

$$x \in]0,1], y \in]0,1], \quad |x-y| \leq \eta, \quad \text{and} \quad \left|\frac{1}{x} - \frac{1}{y}\right| = \frac{\eta_2}{x(x+\eta_2)}.$$

However for $x \in]0,1]$ and $y \in]0,1]$, $|x-y| \leq \eta \Rightarrow \left|\frac{1}{x} - \frac{1}{y}\right| \leq \varepsilon$, thus

$$\frac{\eta_2}{x(x+\eta_2)} \leq \varepsilon,$$

and this for any initial choice of $x \in \left]0, \frac{1}{2}\right[$. Absurd.

2. *Similarly, we consider the function defined as $f(x) = x^2$ on the interval $[0, +\infty[$. We have for x, $y > 0$, $|f(x) - f(y)| = |x^2 - y^2| = |x-y|\,|x+y|$. Let $\varepsilon > 0$. We are going to show that for any $\eta > 0$, we can find x, y verifying $|x-y| \leq \eta$ and $|f(x) - f(y)| > \varepsilon$. As $\mathbb{R}$ is archimedian, and $\eta > 0$ is given, there exists $n \in \mathbb{N}$ such that $n\eta > \varepsilon/2$. Let us then take $x = n$ and $y = x + \eta$. We obtain $|x-y| = \eta$ and $|f(x) - f(y)| = |x-y|\,|x+y| \geq 2n\eta > \varepsilon$.*

These two examples show that for a part open / part closed interval or a closed unbounded interval, the theorem is not verified. It is however important to note that it is not necessary for f to be defined on a closed bounded interval. We previously saw that a Lipschitz function on some interval I is uniformly continuous on I and that $x \mapsto \sqrt{x}$ is uniformly continuous on $[0, +\infty[$ without being Lipschitz.

4.3 Fundamental theorems for differentiable functions

4.3.1 Existence of derivatives

Here is an interesting property for convex functions:

Proposition 4.25. *Let $I =]a,b[$ and $f : I \to \mathbb{R}$ a convex function then f has a left and right derivative at any point $x_0 \in I$*

Proof Let $x_0 \in I$. The function $r_{x_0} : x \mapsto \dfrac{f(x) - f(x_0)}{x - x_0}$ is non decreasing and thus has a left-hand limit and a right-hand limit at x_0. □

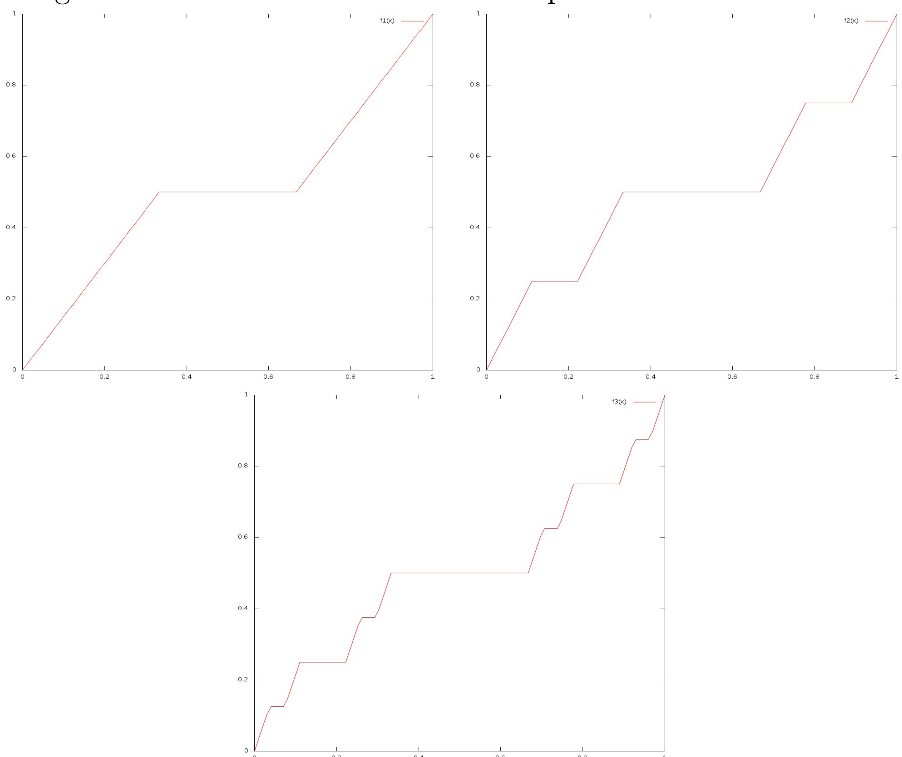
Here is a pleasing result, that extends beyond the bounds of this course (and that does not have to be learnt) that can be rewritten as

Any monotonous function $f : I \to \mathbb{R}$ is differentiable "almost everywhere".

A precisely mathematical meaning to "almost everywhere" exists but will not be given here. Instead let us discuss an example.

Example 21. *We construct a sequence of functions $f_n : [0,1] \to [0,1]$ that are piece-wise linear, but also continuous, non decreasing and such that $f(0) = 0$ et $f(1) = 1$. We choose $f_0(x) = x$ then f_1 piece-wise linear that is worth 0 at 0, 1 at 1 and constant and equal to $1/2$ on $[1/3, 2/3]$. To go from f_{n+1} to f_n, on each interval $[u, v]$ where f_n is not constant, we replace f_n by f_{n+1} that is piece-wise continuous and worth $\dfrac{f_n(u) + f_n(v)}{2}$ on $[(2u+v)/3, (2v+u)/3]$.*

Figure 4.2: first three construction steps of the "devil's stairs"



We can show that this procedure converges and leads to a constant function f on intervals $I_n, n \in \mathbb{N}$ so its derivative exists and is zero. We note that $\sum_{n=0}^{\infty} \mathrm{diam}(I_n) = 1$ which is exactly the length of the interval $[0,1]$. Thus the function f is differentiable except on a "zero-length" set. This function is called **the devil's staircase** *and is differentiable with a derivative worth zero on a* **Cantor set**. *This function shows that the equality*

$$f(1) - f(0) = \int_0^1 f'(s)ds,$$

is only true under certain conditions.

4.3.2 Inverses of continuous and differentiable functions

In this section we study the inverse functions of continuous and sometimes differentiable functions when these are bijective from an interval I to $J = f(I)$. We state below a result on the existence of such a function and its continuity and differentiability. As an example, we focus on the inverse of the tangent function on the interval $I =]-\frac{\pi}{2}, \frac{\pi}{2}[$. Like the inverses of other usual functions, this function plays an important role in maths and modelling (for integral computation for example). For this reason, its properties should be learnt.

Theorem 4.26. *Let $f : I \to \mathbb{R}$ a continuous and strictly monotonous function on I. Then, f is a bijection from I to $J = f(I)$. Thus, there exists a function, denoted f^{-1}, defined on J, whose values are in I such that for any $x \in I$, $y \in J$*

$$x = f^{-1}(y) \Longleftrightarrow y = f(x).$$

In particular we have

$$f^{-1}(f(x)) = x \quad \forall x \in I \quad \text{et} \quad f(f^{-1}(y)) = y \quad \forall y \in J.$$

Furthermore, f^{-1} is continuous on J, and of same monotony as f. If f is odd then f^{-1} is odd. Also, if f is differentiable at $x_0 \in I$ such that $f'(x_0) \neq 0$ then f^{-1} is differentiable at $y_0 = f(x_0) \in J$ and we have

$$\boxed{(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.}$$

Proof The existence of an inverse to f is established in Theorem 4.18. The fact that f^{-1} is continuous when f is is taken as true. So let us suppose that f is differentiable on I.
Let $y_0 \in f(I)$. There exists a unique $x_0 \in I$ such that $y = f(x_0)$. Let us suppose that $f'(x_0) \neq 0$. For any $y \in J$, there exists a unique $x \in I$ such that $y = f(x)$, and we have

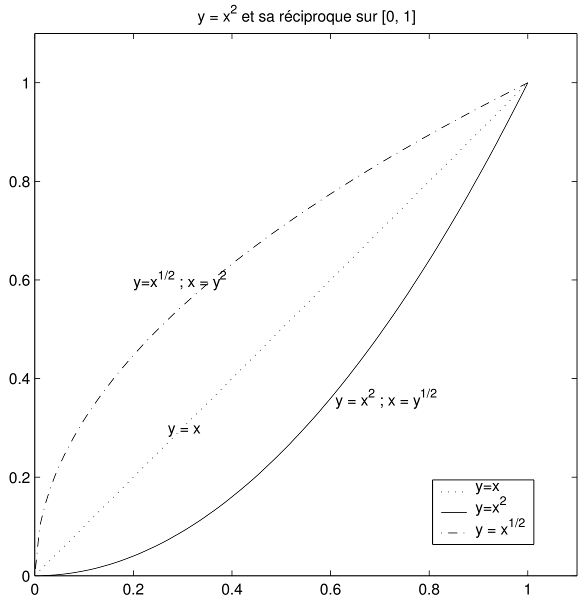
$$\frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \frac{x - x_0}{f(x) - f(x_0)}.$$

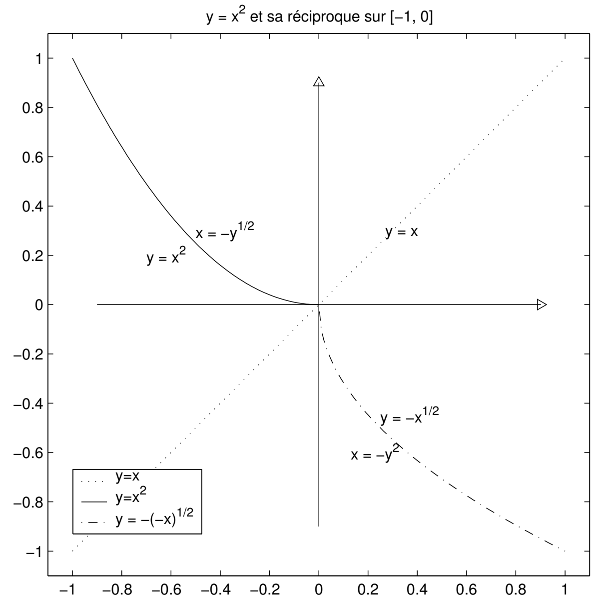
Furthermore, as f^{-1} is continuous, $x - x_0 = f^{-1}(y) - f^{-1}(y_0) \underset{y \to y_0}{\to} 0$, so

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

□

Example 22. *Using the previous theorem, study the differentiability of the function g, inverse to the function $f : \mathbb{R}^+ \to \mathbb{R}^+, x \mapsto x^2$. Give the differentiability of g.*
The graphs of f and g are shown in the left-hand plot below :





Example of an inverse function

The function $x \mapsto \tan x$ is continuous on the intervals $I_k := \left](2\,k-1)\,\dfrac{\pi}{2},\,(2\,k+1)\,\dfrac{\pi}{2}\right[$ for any $k \in \mathbb{Z}$. It is differentiable on each of these intervals and its derivative is defined as

$$\forall x \in I_k,\ \tan'(x) = 1 + \tan^2(x).$$

This derivative is positive, in such a way that the tangent function is monotonous, increasing on each I_k.

Its restriction to the interval $I_0 = \left]-\dfrac{\pi}{2},\,\dfrac{\pi}{2}\right[$ is a continuous bijection from this interval to $J = \mathbb{R}$. The function thus has an inverse function :

Definition 4.27. *We call* **Arctangent** *and denote* $y \mapsto \text{Arctan}\,(y)$ *the function defined on* $\mathbb{R}$ *as*

$$\text{Arctan}\,(y) = x \Longleftrightarrow y = \tan(x) \quad \forall x \in I_0, \quad \forall y \in \mathbb{R}.$$

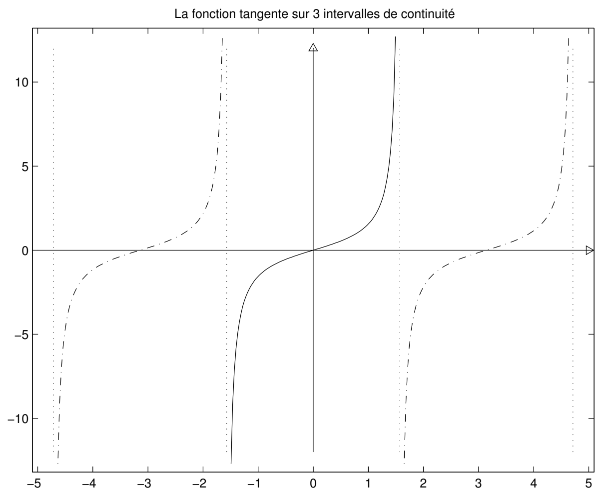
Example 23. *Compute the following values:*

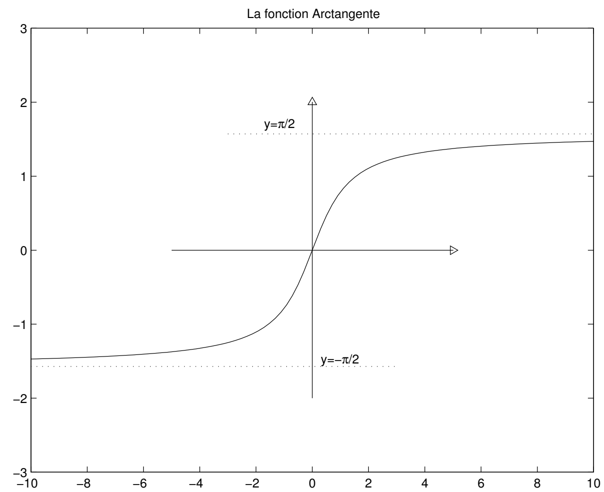
$$\text{Arctan}\,(0) = \quad,\ \text{Arctan}\,(1) = \quad, \text{Arctan}\,(\sqrt{3}) = \quad,$$

$$\tan(\text{Arctan}\,(1)) = \qquad \textit{et}\ \text{Arctan}\,(\tan(\frac{5\,\pi}{6})) = \quad.$$

The figures below show the tangent function on a few intervals where it is defined, and the arctangent function.

The arctangent function is thus odd and increasing, just like the tangent function. It verifies $\tan(\text{Arctan}\,(x)) = x$ for any $x \in \mathbb{R}$, but in general, $\text{Arctan}\,(\tan(x)) \neq x$ for $x \in I_k$. We must recall that the arctangent function is the inverse to the tangent function on I_0 : if $x \in I_k$, for some k in $\mathbb{Z}$, we will then have $\text{Arctan}\,(\tan(x)) = x - k\,\pi$, and for example $\text{Arctan}\,\left(\tan\left(\dfrac{5\,\pi}{6}\right)\right) = -\dfrac{\pi}{6}$.





The tangent function is differentiable on I_0 and $\tan'(x) = 1 + \tan^2(x) \neq 0$ for any $x \in I_0$. Thus, according to the Theorem 4.26, Arctan is differentiable on $\mathbb{R}$ and,

$$\text{Arctan }'(x) = \frac{1}{1+x^2}.$$

At the point $(0,0)$, the curve $y = \text{Arctan }(x)$ will thus have $y = x$ as a tangent.

Furthermore, $\lim_{x\to+\infty} \text{Arctan }(x) = \frac{\pi}{2}$ and $\lim_{x\to-\infty} \text{Arctan }(x) = -\frac{\pi}{2}$ so $y = \frac{\pi}{2}$ and $y = -\frac{\pi}{2}$ are asymptotes to the curve.

Using the same ideas, we can define the functions Arcos and Arcsin that are respectively the inverse function to the restriction of $x \mapsto \cos(x)$ to the interval $[0, \pi]$, and the inverse function to the restriction of $x \mapsto \sin(x)$ to $[-\frac{\pi}{2}, \frac{\pi}{2}]$. These two functions will be studied in tutorial class in exercises 5 and 6 of chapter 8 of the exercise book.

4.3.3 Rolle's theorem

1. Necessary condition for extremum

   Definition 4.28. *We say that $f : I \to \mathbb{R}$ has a* ***local maximum*** *at $a \in I$ if*

   $$\exists V \in \mathcal{V}(a),\ \forall x \in I, \quad x \in V \Rightarrow f(x) \leq f(a).$$

   This maximum if said to be ***strict*** *if*

   $$\exists V \in \mathcal{V}(a),\ \forall x \in I, \quad x \in V \Rightarrow f(x) < f(a).$$

   We define a ***local minimum*** *by replacing $\leq$ by $\geq$ and $<$ par $>$.*
   If f has a local maximum or a local minimum at a, then f is said to have a ***local extremum*** *at a.*
   We say that f has a ***global maximum*** *at a if for any $x \in I$, $f(x) \leq f(a)$. We define a* ***global minimum*** *in the same way. If f has a global maximum or a global minimum at a, then f is said to have a* ***global extremum*** *at a.*

Proposition 4.29. *Let I an open interval, $a \in I$ and $f : I \to \mathbb{R}$ a differentiable function at a. If f has a local extremum at a, then $f'(a) = 0$.*

Beware. The converse is not true. The function defined as $x \mapsto x^3$ is differentiable at 0, its derivative is worth zero at 0, but it does not have any extremum at 0.

Thus the local extrema of a function that is differentiable on I can only be at a point where f' cancels out or at the bounds of I if these belong to I.

2. Rolle's theorem

Theorem 4.30 (Rolle's theorem)**.** *Let $a < b$ two real numbers and let a function $f : [a, b] \to \mathbb{R}$. If f is continuous on $[a, b]$, differentiable on $]a, b[$ and $f(a) = f(b)$ then, there exists $c \in]a, b[$ such that $f'(c) = 0$.*

Proof The result is trivial if the function f is constant. Let us suppose that f is not constant. As f is continuous on $[a, b]$, $f([a, b])$ is an interval of the form $[m, M]$ and $m = \min_{x\in[a,b]} f(x)$ and $M = \max_{x\in[a,b]} f(x)$ which are the only two global extrema of f. Furthermore, as f is non constant, $m \neq f(a)$ or $M \neq f(a)$. Without loss of generality, let us suppose that $M \neq f(a)$. Thus, there exists $c \in]a, b[$ such that $f(c) = M$. Furthermore, as f is differentiable on $]a, b[$, f is differentiable at c and we have :

$$\lim_{h\to 0^+} \frac{f(c+h) - f(c)}{h} = f'(c) = \lim_{h\to 0^-} \frac{f(c+h) - f(c)}{h}.$$

Using the fact that c is a maximum and Theorem 4.10 we deduce from the first equality that $f'(c) \geq 0$, and from the second that $f'(c) \leq 0$, and thus that $f'(c) = 0$. □

Remark 12. *The point c is not necessarily unique as we can see on figure 4.3*

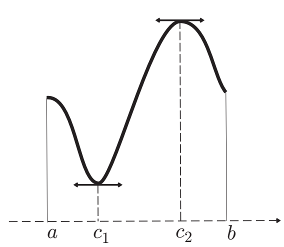


Figure 4.3: Exemple of a function that verifies Rolle's theorem

Example 15. *Application to roots.*

(a) Show that if f is continuous on $[a, b]$, differentiable on $]a, b[$, and if f has n distinct roots then f' has at least $n - 1$ roots that separate those of f.

(b) If f belongs to the $\mathcal{C}^p$ class on $[a, b]$, $p \geq 1$, if f has n different roots on $[a, b]$, and if $p \leq n - 1$, then $f^{(p)}$ has at least $n - p$ roots on $[a, b]$. (Induction on p)

4.3.4 Mean value theorem

Statement

Theorem 4.31 (Mean value theorem)**.**
Let $a < b$ two real numbers and $f : [a,b] \to \mathbb{R}$. If f is continuous on $[a,b]$ and differentiable on $]a,b[$ then there exists $c \in]a,b[$ such that $f(b) - f(a) = f'(c)(b-a)$.

The mean value theorem has a geometrical interpretation that is shown in Figure 4.4.

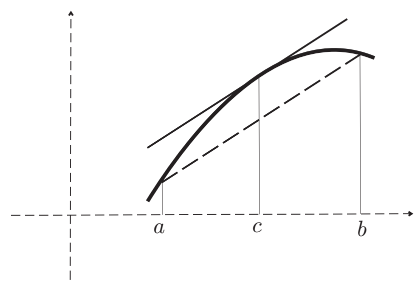


Figure 4.4: Geometrical interpretation of the mean value theorem

Corollary 6 (Mean value inequality)**.**
Let $a < b$ two real numbers, $f : [a,b] \to \mathbb{R}$ continuous on $[a,b]$, differentiable on $]a,b[$ such that f' is bounded on $]a,b[$. We have

$$\forall (x,y) \in [a,b]^2, \quad |f(x) - f(y)| \leq \underset{t \in]x,y[}{sup} |f'(t)||x-y|.$$

Remark 9. *If the function f is of class C^1 on $[a,b]$, then the mean value inequality writes as :*

$$\forall (x,y) \in [a,b]^2, \quad |f(x) - f(y)| \leq \max_{t \in [x,y]} |f'(t)||x-y|.$$

Example 16. *Let $x > 0$. Show that $\dfrac{1}{x+1} < \ln\left(1 + \dfrac{1}{x}\right) < \dfrac{1}{x}$.*

<u>Application</u> : Show that the sequence $(u_n)_{n \in \mathbb{N}^*}$ defined as $u_n = \displaystyle\sum_{k=1}^{n} \frac{1}{k}$ diverges.

For any $k \in \mathbb{N}^*$, we have $\dfrac{1}{k} > \ln\left(1 + \dfrac{1}{k}\right) = \ln(k+1) - \ln k$. We sum for k going from 1 to n, and we obtain $u_n > \displaystyle\sum_{k=1}^{n} (\ln(k+1) - \ln k) = \ln(n+1) \underset{n \to +\infty}{\longrightarrow} +\infty$, thus $\displaystyle\lim_{n \to +\infty} u_n = +\infty$.

Consequences

From this inequality we obtain the following result:

Proposition 4.32. *Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$, differentiable on $]a, b[$. If f' is bounded by k on $]a, b[$, then f is k-Lipschitz on $[a, b]$.*

Proof This is a direct consequence to the mean value inequality.
Suppose that there exists $k \geq 0$ such that $|f'(t)| \leq k$ for any $t \in]a, b[$. Then, for any $x, y \in [a, b]$, $|f(x) - f(y)| \leq \sup_{t \in]x,y[} |f'(t)||x - y| \leq k|x - y|$. □

Example 17. *Let $f : [a, b] \to [a, b]$ of class $\mathcal{C}^1$, $u_0 \in [a, b]$. We suppose that there exists $k \in [0, 1[$ such that $|f'(x)| \leq k$ for any $x \in [a, b]$. We consider the recursive sequence defined by $u_{n+1} = f(u_n)$ for any $n \in \mathbb{N}$. An immediate induction shows us that $(u_n)_{n\in\mathbb{N}}$ is well defined, and that $u_n \in [a, b]$ for any $n \in \mathbb{N}$.*
If $(u_n)_{n\in\mathbb{N}}$ converges to l, then $l = f(l)$ (by continuity of f), and through an induction argument, we obtain that for any $n \in \mathbb{N}$,

$$|u_n - l| \leq k^n |u_0 - l| \leq k^n (b - a),$$

i.e. *$(u_n)_{n\in\mathbb{N}}$ converges to l similarly to k^n, which gives an idea of the convergence speed.*

Application to function monotony

Theorem 4.33. *Let $f : [a, b] \to \mathbb{R}$ continuous on $[a, b]$, differentiable on $]a, b[$. Then,*

1. *the function f is constant on $[a, b]$ if and only if $f' = 0$ on $]a, b[$,*
2. *the function f is non decreasing on $[a, b]$ if and only if $f' \geq 0$ on $]a, b[$,*
3. *the function f is non increasing on $[a, b]$ if and only if $f' \leq 0$ on $]a, b[$,*
4. *if $f' > 0$ on $]a, b[$, then f is increasing on $[a, b]$,*
5. *if $f' < 0$ on $]a, b[$, then f is decreasing on $[a, b]$.*

Remark 13.

- ***Beware !*** *A differentiable function, whose derivative is zero is not necessarily constant. This is true only if this function is defined on an interval. Indeed, the function $f : \mathbb{R}^* \to \mathbb{R}$, $f(x) = 1$ if $x > 0$, $f(x) = -1$ if $x < 0$ is differentiable $\mathbb{R}^*$, and its derivative is zero on $\mathbb{R}^*$. It is constant on each of the intervals on which it is defined.*
- *Points 4. and 5. of the previous theorem are not equivalences. The function $x \mapsto x^3$ is increasing on $\mathbb{R}$ but its derivative at 0 is worth zero.*

Differentiability of an extension

We know that a function which is continuous on an interval I, except at a point a, whose limit ℓ at a is finite, can be extended by continuity at a by setting $f(a) = \ell$. What can we say about differentiability ?

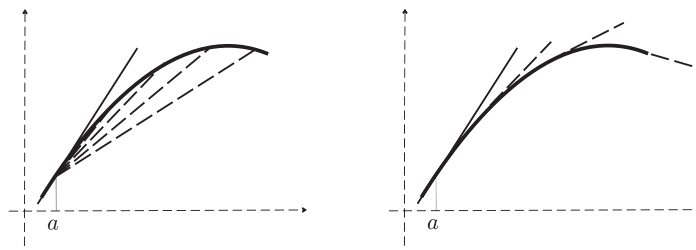


Figure 4.5: Geometrical representation of $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ (left-hand side) and $\lim\limits_{x \to a} f'(x)$ (right-hand side).

The question at hand is therefore as follows : if $f : I \to \mathbb{R}$ is a differentiable function on $I \setminus \{a\}$, what links are there between the existence of $\lim\limits_{x \to a} \dfrac{f(x) - f(a)}{x - a}$ and of $\lim\limits_{x \to a} f'(x)$?

If Figure 4.5, we give a geometrical "representation" of these two limits.

First of all, these two notions are not equivalent, as shown by the following example.

Example 18. *Let*
$$\begin{array}{rcl} f : I & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases} \end{array}.$$
The function f is differentiable at 0, with $f'(0) = 0$. Through theorems on operations, f is differentiable on $\mathbb{R}^*$ and for all $x \in \mathbb{R}^*$, $f'(x) = 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}$. However, $\lim\limits_{x \to 0} 2x \sin \dfrac{1}{x} = 0$ but $\lim\limits_{x \to 0} \cos \dfrac{1}{x}$ does not exist so $\lim\limits_{x \to 0} f'(x)$ does not exist.

However, if f' has a finite limit at a then f is differentiable at a. More precisely,

Theorem 4.34. *Let I an interval of $\mathbb{R}$ containing more than one point $a \in I$. We consider a function $f : I \setminus \{a\} \to \mathbb{R}$ such that*

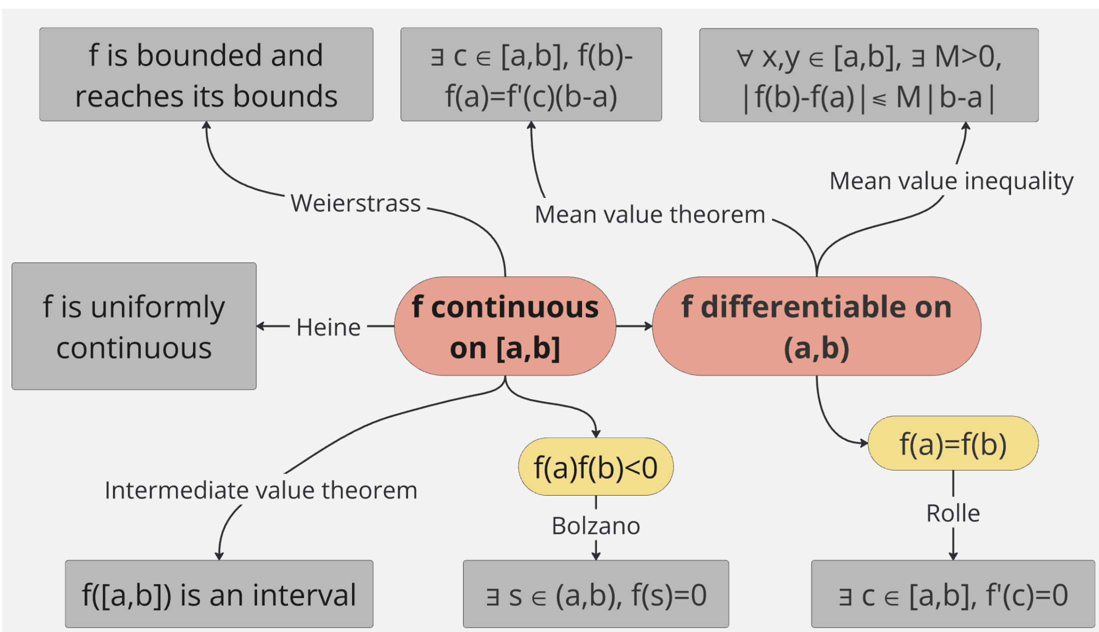
- $\lim\limits_{x \to a} f(x) = L$ *exists,*
- *f is differentiable on $I \setminus \{a\}$ and $\lim\limits_{x \to a} f'(x) = \ell$ exists.*

Then the function
$$\begin{array}{rcl} \tilde{f} : I & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} f(x) & \text{if } x \neq a \\ L & \text{if } x = a \end{cases} \end{array}$$
is continuous and differentiable at a with $\tilde{f}'(a) = \ell$.

Furthermore, if f' is continuous on $I \setminus \{a\}$ then $\tilde{f}$ is of class $\mathcal{C}^1$ on I.

Proof The function $\tilde{f}$ is continuous on I and differentiable on $I \setminus \{a\}$. Let us now show that $\tilde{f}$ is differentiable at a. Let $x \in I \setminus \{a\}$. As $\tilde{f}$ verifies the mean value theorem assumptions on $[a, x]$ (or $[x, a]$), there exists $c_x \in (a, x)$ (or (x, a)) such that $\tilde{f}(x) - \tilde{f}(a) = \tilde{f}'(c_x)(x - a)$, *i.e.* $\dfrac{\tilde{f}(x) - \tilde{f}(a)}{x - a} = \tilde{f}'(c_x)$.
As c_x is between a and x, $\lim\limits_{x \to a} c_x = a$, so $\lim\limits_{x \to a} \tilde{f}'(c_x) = \lim\limits_{y \to a} \tilde{f}'(y) = \ell$. We deduce that $\lim\limits_{x \to a} \dfrac{\tilde{f}(x) - \tilde{f}(a)}{x - a} = \ell$, *i.e.* $\tilde{f}$ is differentiable at a and its derivative is ℓ. Moreover, if f' is continuous on $I \setminus \{a\}$, as $\tilde{f}'(a) = \ell = \lim\limits_{x \to a} f'(x)$, $\tilde{f}$ is of class $\mathcal{C}^1$ on I. □

Remark 14. *If $\lim\limits_{x \to a} f'(x) = \pm\infty$, the same reasoning proves that the curve of f has a vertical tangent at a, and so that f is not differentiable at a.*
If $\lim\limits_{x \to a} f'(x)$ does not exist in $\bar{\mathbb{R}}$, we use Newton's difference quotient to study the differentiability of f.



4.4 Recap quiz

1. Give a function f whose definition domain is $\mathcal{D}_f$ and such that

$$\forall x \in \mathcal{D}_f, f(x) > 0 \text{ and } \lim_{x \to +\infty} f(x) = 0.$$

2. Draw the graph of a function defined on $\mathbb{R}$ and that has jump discontinuities at points 1 and 5.

3. Let f be a function defined on $\mathbb{R}^*$. Give a necessary and sufficient condition for f to be extendable through continuity on $\mathbb{R}$.

4. Draw the graph of a function defined on $[-1;1]$ and piecewise continuous on this interval.

5. Draw the graph of a function defined on $[-1;1]$ and not piecewise continuous on this interval.

6. Let f a bijective and differentiable function on $\mathbb{R}$. Give the differentiability domain and the derivative of the inverse function f^{-1}.

7. Use a drawing to explain why the inverse of a differentiable function f is not differentiable at the points $y_0 = f(x_0)$ where $f'(x_0) = 0$.

8. Show that the function Arccos (defined in exercise 5 of chapter 8 of the exercise book) is continuous on $[-1,1]$, differentiable on $]-1,1[$ and that

$$\text{Arccos}'(x) = -\frac{1}{\sqrt{1-x^2}} \quad \forall x \in]-1,1[.$$

9. Show that the function $g : x \longmapsto \sqrt{\cos x}$ is continuous and bijective from $[0, \frac{\pi}{2}]$ to $[0,1]$. Study the differentiability of g^{-1}. Compute the function g^{-1} and its derivative.

10. State the mean value theorem.

11. Let a and b two reals such that $a < b$ and f defined and continuous on $[a;b]$. Give all the properties that hold for f.

12. Let I an interval of $\mathbb{R}$. What do the two following properties mean and what logical link exists between them?

    (a) $\forall \epsilon > 0, \forall x \in I,\ \exists \eta > 0,\ \forall y \in I, |x-y| < \eta \Rightarrow |f(x) - f(y)| < \epsilon$.

    (b) $\forall \epsilon > 0,\ \exists \eta > 0,\ \forall x \in I, \forall y \in I, |x-y| < \eta \Rightarrow |f(x) - f(y)| < \epsilon$.

13. State Rolle's theorem.

14. We consider the function defined for all $x \in \mathbb{R}$ as $f(x) = \dfrac{4}{\pi}\text{Arctan}\, x - x$. Show, <u>without computing it</u>, that the derivative of f is worth zero at least twice on $[-1;1]$.

15. Let $f : [a,b] \to \mathbb{R}_+^*$ continuous and differentiable on $[a,b]$. Show that there exists $c \in]a,b[$ such that $\ln\left(\dfrac{f(b)}{f(a)}\right) = \dfrac{f'(c)}{f(c)}(b-a)$.
    Hint : Use the function $\ln(f)$.

16. Let a, b two reals and f, a function of class $\mathcal{C}^1$ on $\mathbb{R} \setminus \{a\}$. We define

$$\tilde{f}(x) = \begin{cases} f(x) & \text{if} \quad x \neq a \\ b & \text{if} \quad x = a \end{cases}$$

(a) We suppose that $\lim_{x \to a} f(x) = b \in \mathbb{R}$. What can we deduce on the function $\tilde{f}$?

(b) We suppose that $\lim_{x \to a} f'(x) = L \in \mathbb{R}$. What can we deduce on the function $\tilde{f}$? What theorem leads us to this conclusion?

(c) We suppose that $\lim_{x \to a} f'(x) = +\infty$. What can we deduce on the function $\tilde{f}$?

(d) We suppose that the function f' does not have a limit at a. What can we deduce on the function $\tilde{f}$?

(e) Application : We set $a = 0$ and $f(x) = x^2 \ln(x^2)$. Determine the real number b that makes the function $\tilde{f}$ continuous on $\mathbb{R}$. Is the function $\tilde{f}$ of class $\mathcal{C}^1$ on $\mathbb{R}$? of class $\mathcal{C}^2$ on $\mathbb{R}$?

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