EXERCISE SHEET - MATH 1

1 Use of the $\sum$ sign

The aim of this section is to introduce the $\sum$ notation and its main properties (permutations, change of index, telescopic sums ...)
Let $m, n \in \mathbb{Z}$ such that $m \leq n$. We introduce the notation $[\![m; n]\!]$ which translates as the set of the intergers between m and n i.e.

$$[\![m; n]\!] := \{m, m+1, \ldots, n-1, n\} = \{k \in \mathbb{Z} \mid m \leq k \leq n\}.$$

We recall that $\text{card}\,([\![m; n]\!]) = n - m + 1$.

Definition 1.1. *Let $I = [\![m, n]\!]$ with $m, n \in \mathbb{Z}$ and $m \leq n$, and let $(x_i)_{i \in I}$ a family od real or complex numbers indexed by I. We denote $\sum_{k=m}^{n} x_k$ or $\sum_{m \leq k \leq n} x_k$ the sum of elements in the family $(x_i)_{i \in I}$. The expression $\sum_{k=m}^{n} x_k$ reads as "the sum for k from m to n os x indexed by k".*
Mathematically this notation designates the sum $x_m + x_{m+1} + \cdots = x_n$.

Remark 1.

1. The number $\sum_{k=m}^{n} x_k$ depends on m and n, **it does not depend on** k.
2. The index k in the sum is called the *dummy variable*. The choice of summation index is completely arbitrary, we could choose any letter except of course those that already have meaning. For example, writing $\sum_{n=0}^{n} x_n$ does not make sense, as n appears as both a bound on the sum's index and as the sum's dummy variable.

Exercise 1.1 ($\star$)**.**

1. Compute $A = \sum_{i=0}^{6} 1$.
2. Rewrite the following sums using symbolic notations

   (a) $S_n = \ln(2) + \ln(3) + \cdots + \ln(n)$;

   (b) $T_p = 1 + 2^3 + \cdots + p^3$;

   (c) $B = 2^3 + 4^3 + \cdots + 100^3$;

   (d) $C = 1^3 + 3^3 + \cdots + 101^3$;

   (e) $D = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$.

   (f) $E = \cos\sqrt{2} + \cos 2\sqrt{2} + \cdots + \cos 9\sqrt{2}$

   (g) $F = 2 + \frac{1}{2} + \frac{2}{9} + \frac{1}{8} + \frac{2}{25} + \cdots + \frac{2}{81}$

Proposition 1.1 (Linearity of the sums)**.** *Let $m, n \in \mathbb{Z}$ such that $m \leq n$. Let $a_m, a_{m+1}, \ldots, a_n$, $b_m, b_{m+1}, \ldots, b_n$ be any real or complex numbers. We have that*

$$\sum_{k=m}^{n} (a_k + b_k) = \sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k$$

and for any real or complex λ

$$\sum_{k=m}^{n} \lambda a_k = \lambda \sum_{k=m}^{n} a_k$$

Exercise 1.2 ($\star$)**.**

Let x_0, x_1, x_2, x_3 and x_4 be any real or complex numbers and $B = \sum_{k=0}^{3} x_k$. Express the following sums as functions of B.

$$C = \left(\sum_{k=0}^{3} x_k \right) + 1, \quad D = \sum_{k=0}^{3} (x_k + 1), \quad E = \sum_{k=0}^{3} x_{k+1}$$

Remark 2. In general,

$$\sum_{k=m}^{n} a_k b_k \neq \left(\sum_{k=m}^{n} a_k \right) \left(\sum_{k=m}^{n} b_k \right)$$

Futhermore, as the summation is associative and commutative, we can group our sums into smaller and/or sum in the order that we want.

Proposition 1.2 (Chasles relation)**.** *For any $j \in [\![m; n]\!]$, we have*

$$\sum_{k=m}^{n} a_k = \sum_{k=m}^{j} a_k + \sum_{k=j+1}^{n} a_k$$

One very useful technique for dealing ith sums is the change of index. This is detailed in the following Proposition.

Proposition 1.3 (Change of index). *Let m, p and n be three positive integers such that $m \leq n$.*
Let $a_{p+m}, \ldots, a_{n+m}$ be real or complex numbers. Then

$$\sum_{k=p}^{n} a_{k+m} = \sum_{j=p+m}^{n+m} a_j.$$

It is said that we have performed the change of index $j = k + m$ in the sum $\sum_{k=p}^{n} a_{k+m}$.

Remark 3. Once we become used to these manipulations, we simply write that

$$\sum_{k=p}^{n} a_{k+m} = \sum_{k=p+m}^{n+m} a_k$$

and specify that we have performe the change of index $k \leftarrow k + m$.

Exercise 1.3 ($\star$).
Let $n \in \mathbb{N}^*$. Perform a change of index in

1. the sum $\sum_{k=2}^{10} \frac{1}{(k-1)^2}$ to sum $\frac{1}{k^2}$ terms ;
2. the sum $\sum_{k=-4}^{100} \frac{k}{k+5}$ so that the sum begins at index 0 ;
3. the sum $\sum_{k=4}^{n+2} \frac{x^{k-3}}{(k-3)^2}$

Using both the linearity of the sums and a change of index we show the following property.

Proposition 1.4 (Telescopic sums). *Let $m, n \in \mathbb{Z}$ such that $m \leq n$. Let $a_m, a_{m+1}, \ldots, a_n$ be any real or complex numbers. Then*

$$\sum_{k=m}^{n} (a_{k+1} - a_k) = a_{n+1} - a_m$$

Exercise 1.4 ($\star\star$).

1. Compute $\sum\limits_{k=1}^{n} \ln\left(\frac{k}{k+1}\right)$
2. Prove that
$$\sum_{k=1}^{n} k \cdot k! = (n+1)! - 1$$
(Hint : use the famous mathematical magic trick $k = (k+1) - 1$)

Proposition 1.5 (Summing the n first positive integers)**.** *Let $n \in \mathbb{N}^*$, then*
$$S_n := 1 + 2 + \cdots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Exercise 1.5 ($\star\star\star$).

1. Let $x \mapsto P(x)$ be a function : give a simple formula for the sum
$$\sum_{k=0}^{n} \left(P(k+1) - P(k)\right).$$
2. We set $P(x) = x^2$. Give two different expressions for the sum A defined as
$$A := \sum_{k=0}^{n} \left((k+1)^2 - k^2\right).$$
Deduce $S_n^{(1)} := \sum\limits_{k=0}^{n} k$.
3. We set $P(x) = x^3$. Give two different expressions for the sum B defined as
$$B := \sum_{k=0}^{n} \left((k+1)^3 - k^3\right).$$
Deduce $S_n^{(2)} := \sum\limits_{k=0}^{n} k^2$.
4. Use the strategy given in this exercise to compute $S_n^{(3)} := \sum\limits_{k=0}^{n} k^3$.

Exercise 1.6 ($\star\star$).
Compute the following sums
$$A = \sum_{k=0}^{n}(k+1), \quad B = \sum_{k=1}^{n}(2k+1), \quad C = \sum_{k=3}^{n+4}(k-2) \quad \text{and} \quad D = \sum_{k=1}^{n}(nk-1).$$

One of the most well-known sum is the geometric sum defined in the following property.

Proposition 1.6 (Sum of the terms in a geometric sum)**.** *Let q be a real or complex number and n a positive integer then :*

1. *If $q \neq 1$,*

$$\sum_{k=0}^{n} q^k = \frac{q^{n+1}-1}{q-1}.$$

2. *If $q = 1$,*

$$\sum_{k=0}^{n} q^k = \sum_{k=0}^{n} 1 = n+1.$$

Exercise 1.7 ($\star\star$)**.**
Compute the following sums

$$A = \sum_{k=0}^{n} 2^k, \quad B = \sum_{k=10}^{n+5} 2^k, \quad C = \sum_{k=0}^{2n-1} 2^{\frac{k}{2}} \quad \text{and} \quad D = \sum_{k=0}^{n} 2^{2k-1}.$$

Proposition 1.7. *For any integer n and any two real or complex numbers a and b we have the following formula*

$$a^{n+1} - b^{n+1} = (a-b) \times \sum_{k=0}^{n} a^k b^{n-k}.$$

Exercise 1.8 ($\star\star\star$)**.**

1. Prove the previous proposition by expanding the right hand terme and using a change of index.
2. Use the formula from the proposition to compute, for any $x \in \mathbb{R} \setminus \{1\}$, $\sum\limits_{k=0}^{n} x^k$.
3. Compute the following sums

$$A = \sum_{k=0}^{n} 2^k 3^{10+n-k} \quad \text{and} \quad B = \sum_{k=0}^{n} (-1)^k 2^{n-k}$$

Exercise 1.9 ($\star\star\star$)**.**
Newton's binomial

1. Use an induction reasoning to prove that for any $(a,b) \in \mathbb{C}^2$ and any $n \in \mathbb{N}$,

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k} = \sum_{k=0}^{n} C_n^k a^k b^{n-k}.$$

2. Let $n \in \mathbb{N}^*$. For any real number x, we define

$$f(x) = (1+x)^n.$$

   (a) Write $f(x)$ as a sum of powers of x.
   (b) Compute the derivative of f at $x = 1$ and its integral over $[0,1]$ using its definition and using its power expansion.
   (c) Deduce the value of the following sums :

$$D_n = \sum_{k=0}^{n} k\binom{n}{k} \quad \text{and} \quad I_n = \sum_{k=0}^{n} \frac{1}{k+1}\binom{n}{k}.$$

   (d) Adapt this method to compute $\sum_{k=0}^{n} k(k-1)\binom{n}{k}$. Deduce the value of $\sum_{k=0}^{n} k^2\binom{n}{k}$.

Exercise 1.10 ($\star\star\star$)**.**
For any $n \in \mathbb{N}$ and $x \in \mathbb{R}$, let $C_n(x)$ and $S_n(x)$ be defined as follows :

$$C_n(x) = \sum_{k=0}^{n} \cos(kx) \quad \text{and} \quad S_n(x) = \sum_{k=0}^{n} \sin(kx)$$

1. Compute C_n and S_n when x is a multiple of 2π.
2. Suppose that x is not a multiple of 2π. By computing $U_n(x) = C_n(x) + \boldsymbol{i}\, S_n(x)$ find the value of $C_n(x)$ and $S_n(x)$. (Here $\boldsymbol{i}$ stands for the complexe number such that $\boldsymbol{i}^2 = -1$)

Definition 1.2. *Let $I = [\![m;n]\!] \times [\![p;q]\!]$ with $m,n,p,q \in \mathbb{Z}$ and $m \le n$, $p \le q$ and $(x_{k\ell})_{(k,\ell)\in I}$ a family of real or complex numbers indexed by I. The sum of term of the family $(x_{k\ell})_{(k,\ell)\in I}$ is written as*

$$\sum_{k=m}^{n}\sum_{\ell=p}^{q} x_{k\ell} = \sum_{\substack{m\le k\le n\\ p\le \ell\le q}} x_{k\ell}$$

Proposition 1.8 (Permutation of sums)**.** *Let $(x_{i,j})_{1\le i,j\le n}$ be a family of real or complexe numbers, then :*

$$\sum_{1\le i,j\le n} x_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{n} x_{ij}$$

$$\sum_{1\le i\le j\le n} x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{j} x_{ij} = \sum_{i=1}^{n}\sum_{j=i}^{n} x_{ij}$$

$$\sum_{1\le i<j\le n} x_{ij} = \sum_{j=1}^{n}\sum_{i=1}^{j-1} x_{ij} = \sum_{i=1}^{n}\sum_{j=i+1}^{n} x_{ij}$$

Exercise 1.11 ($\star\star$)**.**
Compute the following sums

$$A = \sum_{i=1}^{n} \sum_{j=1}^{n} ij \quad \text{and} \quad B = \sum_{1 \leq i \leq j \leq n} ij$$

Exercise 1.12 ($\star\star\star$)**.**
Let $n \in \mathbb{N}$. Compute the coefficient of the polynomial

$$P(x) = \sum_{k=0}^{n} (1+x)^k.$$

2 Linear second order differential equations with constant coefficients

Exercise 2.1 ($\star$)**.**
Compute the solutions to the differential equation :

$$y''(x) - 4y'(x) + 3y(x) = g(x),$$

in each of the following cases :

1. $g(x) = x + 1$.
2. $g(x) = e^{2x}$.
3. $g(x) = e^{x}$.
4. $g(x) = 2(x + 1) + 3e^{x}$.

Exercise 2.2 ($\star$)**.**
Compute the solutions to the differential equation :

$$y''(x) + 2y'(x) + 4y(x) = g(x),$$

in each of the following cases :

1. $g(x) = 3$.
2. $g(x) = xe^{-2x}$.

Exercise 2.3 ($\star$)**.**

1. Compute the solution to $y''(x) + 6y'(x) + 9y(x) = xe^{-3x}$ with $y(0) = y'(0) = 1$.
2. Compute the set of solutions to $y''(x) + 6y'(x) + 9y(x) = x$.
3. Compute the set of solutions to $y''(x) + 6y'(x) + 9y(x) = 9x + 2xe^{-3x}$.

Exercise 2.4 ($\star\star$)**.**
We consider a masse m hanging on a spring and immersed in a fluid. We suppose that the mss is subject to a unique vertical displacement. We denote as x the algebraic displacement with respect to the equilibrium position. We denote as k the spring's stiffness coefficient ans as α the fluid's friction coefficient. These are both considerer to be positive.
Forces acting on mass m :

- the spring's tension : $\vec{T} = -kx\vec{e_x}$,
- friction due to the fluid : $\vec{F} = -\alpha\dot{x}\vec{e_x}$.

Applying the fondamental principle of dynamics, it can be shown that the differential equation verified by x is

$$m\ddot{x} + \alpha\dot{x} + kx = 0.$$

1. Solve the equation and give the solutions as a function of the parameters α, k and m.
2. Give a physical interpretation of the observed behavior.

3 Polynomial functions and rational fractions

3.1 Polynomial functions

Exercise 3.1 ($\star$).
Give the quotient and the remainder of the euclidean division of polynomial A by polynomial B in the following cases.

1. $A(x) = x^7 - 2x^6 + 3x^4 - 2x + 1$; $B(x) = x^3 + 1$.
2. $A(x) = x^4 + \boldsymbol{i}\, x^3 + 3x - 1$; $B(x) = x^2 + (1 + \boldsymbol{i}\,)x + 1$.

Exercise 3.2 ($\star$).
Find $P \in \mathbb{R}[x]$ such that :

1. $\deg(P) = 6$, 1 is a double root, 3 is a triple root and $P(0) = 1$.
2. $\deg(P) = 7$, $\boldsymbol{i}$ is a simple root, $1 - \boldsymbol{i}$ is a double root, 3 is a simple root and $P(1) = 2$.

Exercise 3.3 ($\star\star$).

1. Factorise, without using a discriminant, the following polynomials (look for trivial roots).

   $P_1(x) = x^2 + 9, \quad P_2(x) = x^2 - 4x + 3, \quad P_3(x) = x^2 + 6x + 8, \quad P_4(x) = x^2 - 81.$

2. (a) Find $P \in \mathbb{R}[x]$ of degree 2 whose roots α and β verify

   $$\alpha + \beta = 2 \quad \text{and} \quad \alpha\beta = 6.$$

   (b) Compute α et β.
3. Let $Q(x) = x^2 + 10x - 1$. Show (without computing the roots) that Q has two real roots, whose signs are different.
4. Let $R(x) = x^2 - 2x + 10$. Show (without computing the roots) that R has two complex roots that are conjugates and whose real parts are positive.

Exercise 3.4 ($\star\star$).
Let $P \in \mathbb{R}[x]$ defined as $P(x) = x^4 + 2x^3 + ax^2 + bx + 36$ with a and b two real numbers.

1. Find (without computing the quotient) the remainder of the Euclidean division of P by $(x - 1)$.
2. Find (without computing the quotient) the remainder of the Euclidean division of P by $(x^2 - 1)$.
3. Show that there exists two real numbers a and b such that P has two double roots belonging to $\mathbb{Q}$ (use the relations between coefficients and roots).

Exercise 3.5 ($\star$).
Let $A(x) = x^5 + 3x^4 = 2x^3 - 2x^2 - 3x - 1$.

1. Show that -1 is a multiple root of A and find its multiplicity.
   (Hint : compute the derivatives of A)
2. Factorise A into a product of irreducible polynomials in $\mathbb{R}[x]$.

Exercise 3.6 ($\star\star$).
Let $P(x) = x^5 + 2x^4 + 6x^3 + 8x^2 = 8x$.

1. Check that 0 and $2\boldsymbol{i}$ are roots of P.
2. Deduce from the previous questions that $B(x) = x^3 + 4x$ divides P.
3. Compute the Euclidean division of P by B.
4. Factorise P into a product of irreducible polynomials in $\mathbb{R}[x]$ and then in $\mathbb{C}[x]$.

Exercise 3.7 ($\star\star\star$).

1. Facorise $P(x) = x^3 - 1$ and $Q(x) = x^4 + 4$ in $\mathbb{C}[x]$ and in $\mathbb{R}[x]$.
2. Let $P_n(x) = x^n - 1$, $n \in \mathbb{N}^*$.
   (a) Compute the roots of P_n in $\mathbb{C}$.
   (b) Deduce the factorisation of P_n in $\mathbb{C}[x]$ and then in $\mathbb{R}[x]$.
   (Hint : use the parity of n)

3.2 Polynomial functions : extension exercises

Exercise 3.8 ($\star\star$).
Let $P \in \mathbb{R}[x]$, $P(x) = x^4 - 5x^3 + ax^2 + bx - 10$.

1. Show that there exists two real numbers a and b such that -1 and 2 are the two only real roots of P.
2. Compute the other roots of P.

Exercise 3.9 ($\star\star$).
Let $n \in \mathbb{N}^*$. Find the remainder of the euclidean division of $x^{2n}+x^n+1$ by $(x-1)(x+1)$.

Exercise 3.10 ($\star\star$).
Let $P(x) = x^4 + x^3 + 5x^2 + 4x + 4$.

1. Let $\boldsymbol{j} = -\frac{1}{2} + \boldsymbol{i}\frac{\sqrt{3}}{2}$. Compute $\boldsymbol{j}^3$ and $1 + \boldsymbol{j} + \boldsymbol{j}^2$.
2. Show that $\boldsymbol{j}$ is a root of P.
3. Deduce, without calculations, that $Q(x) = x^2 + x + 1$ divides P.

4. Perform the euclidean division of P by Q.
5. Factorise P into a product of irreducible polynomial in $\mathbb{R}[x]$ and then $\mathbb{C}[x]$.

Exercise 3.11 ($\star\star\star$)**.**
Let P be a polynomial of $\mathbb{R}[x]$, $\alpha = a + \boldsymbol{i}\, b \in \mathbb{C}$ with $b \neq 0$ and $B(x) = x^2 - 2\mathrm{Re}(\alpha)x + |\alpha|^2$.

1. Show that B divides P if and only if α is a root of P.
2. Let $n \in \mathbb{N}$.
   (a) Compute $\boldsymbol{i}^{\,n}$ and $\boldsymbol{j}^{\,n}$ where $\boldsymbol{j} = e^{\frac{2\boldsymbol{i}\pi}{3}}$ is a cubic root of unity.
   (b) Let $B(x) = x^2 + 1$ and $P(x) = x^n + 1$. For which values of n does B divides P ?
   (c) Same question with $B(x) = x^2 + x + 1$ and $P(x) = x^n - 1$.

3.3 Rational functions

Exercise 3.12 ($\star$)**.**
We consider the rational functions :

$$F_1(x) = \frac{x^4 - x + 1}{x^3 - 1} \quad \text{and} \quad F_2(x) = \frac{1}{(x+1)^2(x^2+x+4)}.$$

1. Study of F_1.
   (a) Is F_1 irreducible ?
   (b) Give the whole part of F_1.
   (c) Give the poles of F_1 in $\mathbb{C}[x]$ and then in $\mathbb{R}[x]$. Specify their multiplicity.
   (d) Give the definition domain of F_1 in $\mathbb{C}$ and then in $\mathbb{R}$.
   (e) Decompose F_1 into partial fractions in $\mathbb{R}$ and in $\mathbb{C}$.
2. Same questions for F_2.

Exercise 3.13 ($\star$)**.**
Decompose the following rational functions into partial fractions in $\mathbb{R}$:

$$G_1(x) = \frac{x^3}{x^2 + 2x + 10}, \quad G_2(x) = \frac{2x - 6}{(x^2 - 3x + 2)^2}, \quad G_3(x) = \frac{12x^2 + 8x - 4}{(x^2 - 1)^2}$$

$$\text{and} \quad G_4(x) = G_3(x^2) = \frac{12x^4 + 8x^2 - 4}{(x^4 - 1)^2}.$$

Exercise 3.14 ($\star\star$).

1. Decompose the following function into partial fractions on $\mathbb{R}$:

$$F(x) = \frac{1}{x(x+1)(x+2)}.$$

2. Deduce for $n \geq 4$, the value of s_n and then $\lim_{n\to+\infty} s_n$, where

$$s_n = \sum_{k=4}^{n} F(k) = \sum_{k=4}^{n} \frac{1}{k(k+1)(k+2)}.$$

Exercise 3.15 ($\star\star$).

1. Decompose the following rational functions into partial fractions on $\mathbb{R}$:

$$H_1(x) = \frac{4x}{(x+1)(x^2+1)}, \quad H_2(x) = \frac{4x}{(x+1)^2(x^2+1)}, \quad H_3(x) = \frac{4x}{(x+1)(x^2+1)^2}$$

$$\text{and} \quad H_4(x) = \frac{4x}{(x+1)^2(x^2+1)^2}.$$

2. Let the rational function $H(x) = \frac{1+2x-x^2}{(x+2)^4}$.
   (a) Compute the partial fraction decomposition in $\mathbb{R}$ of $H(t-2)$.
   (b) Deduce the partial fraction decomposition in $\mathbb{R}$ of H.

4 Limits, continuity and differentiability

Exercise 4.1 ($\star$)**.**
We consider a function f defined on a neighborhood of point a (except perhaps at point a). What do the following statements mean ?

1. $\exists \ell \in \mathbb{R}, \exists \varepsilon > 0, \exists \eta > 0 \ : \ \forall x \in \mathcal{D}_f, \ |x - a| \leq \eta \Rightarrow |f(x) - \ell| \leq \varepsilon.$
2. $\exists \ell \in \mathbb{R}, \forall \varepsilon > 0, \exists \eta > 0 \ : \ \forall x \in \mathcal{D}_f, \ |x - a| \leq \eta \Rightarrow |f(x) - \ell| \leq \varepsilon.$
3. $\exists \ell \in \mathbb{R}, \exists \eta > 0, \forall \varepsilon > 0 \ : \ \forall x \in \mathcal{D}_f, \ |x - a| \leq \eta \Rightarrow |f(x) - \ell| \leq \varepsilon.$

Exercise 4.2 ($\star$)**.**

1. Let a and b be two real numbers. Show that
$$||a| - |b|| \leq |a - b|.$$
2. Let f be a function that tends to ℓ when x tends to a. Show that the function $|f|$ tends to $|\ell|$ when x tends to a.

Exercise 4.3 ($\star\star$)**.**

1. Show that for any $a \in \mathbb{R}$, any $b \in \mathbb{R}$, we have
$$\min(a, b) = \frac{a + b - |a - b|}{2}.$$
2. Let $f, g : I \to \mathbb{R}$ be two differentiable functions on the interval I.
Let $\mathcal{N} = \{x \in I \mid f(x) = g(x)\}$. We set for any $x \in I$, $h(x) = \min(f(x), g(x))$.
   (a) Is the function h continuous on $I \setminus \mathcal{N}$? Is it differentiable on $I \setminus \mathcal{N}$?
   (b) Is the function h continuous on I ?
   (c) Let $x_0 \in \mathcal{N}$. Study the differentiability of h at x_0.
   (Hint : We could give two examples)

Exercise 4.4 ($\star\star$)**.**
Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function at 0 such that
$$\forall x \in \mathbb{R}, f(2x) = f(x).$$

1. We fix $x \in \mathbb{R}$. Prove that for all $n \in \mathbb{N}$, $f(x) = f\left(\frac{x}{2^n}\right)$.
2. Show that f is constant.
(Hint : take the limit when n goes to $+\infty$ in the previous equality)

Exercise 4.5 ($\star$).

1. Compute the limit at $x = 1$ of :

   (a) $f(x) = \dfrac{\ln(x)}{x-1}$; (b) $g(x) = \dfrac{\ln(x)}{x^5 - 1}$

2. Compute the limit at $x = 0^+$ of :

   (a) $f(x) = \ln\left(\dfrac{1}{x^2}\right)$, (b) $g(x) = \dfrac{\sqrt{x}}{\sqrt{x^2 + \sin(x)}}$

3. Compute the limit when x tends to $+\infty$.

   (a) $f_1(x) = (x - \ln(|x|))\, e^x$,

   (b) $f_2(x) = \dfrac{\sqrt{x}}{\sqrt{x^2 + \sin(x)}}$,

   (c) $f_3(x) = \dfrac{\sin(x)}{x^4}$,

   (d) $f_4(x) = \dfrac{x^\alpha + 2}{x^3 + 1}$, $\alpha \in \mathbb{R}_+^*$,

   (e) $f_5(x) = \dfrac{x^3 \ln(x^4)}{e^x}$,

   (f) $f_6(x) = \dfrac{8x^7 + x^3}{(2x+1)^7 - \ln(x^5)}$,

   (g) $f_7(x) = \left(\sqrt{x}\right)^{\frac{1}{3\ln(x)}}$.

Exercise 4.6 ($\star\star$).
We consider the four real functions defined below :

$$f_1 : x \mapsto \sqrt{1 - e^x}, \quad f_2 : x \mapsto \ln(x^2 - 1), \quad f_3 : x \mapsto \frac{1}{x-1} \quad \text{and} \quad f_4 = f_3 \circ f_2.$$

1. Study the definition, continuity and differentiability domains of these functions.
2. Compute the following limits (if they exist) :

$$\lim_{x \to -\infty} f_1(x), \quad \lim_{x \to 1^+} f_2(x) \quad \text{and} \quad \lim_{x \to a} f_4(x) \quad \text{where} \quad a = \sqrt{1 + e}.$$

3. Compute the derivatives of these functions.

Exercise 4.7 ($\star\star\star$).
Let the functions $f(x) = \ln(1 + x)$ and $g(x) = \frac{1}{1+x}$.

1. Compute the definition domain of f and g.
2. Do these functions belong to the $\mathcal{C}^\infty$ class on their definition domain ?
3. Let $n \in \mathbb{N}$. Compute the n'th derivative of the function f.

5 Fundamental analysis techniques

5.1 Study of hyperbolic functions

In this section, we define three new functions called the hyperbolic functions. The **hyperbolic cosine** and **hyperbolic sine** functions are the functions defined for any $x \in \mathbb{R}$ as

$$\cosh(x) = \text{ch}(x) = \frac{e^x + e^{-x}}{2} \quad \text{and} \quad \sinh(x) = \text{sh}(x) = \frac{e^x - e^{-x}}{2}.$$

We called them sine and cosine due to the similarity between their definitions and the Euler formulae. We also define the **hyperbolic tangent** function as :

$$\forall x \in \mathbb{R}, \tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}.$$

Exercise 5.1 ($\star$)**.**

1. Check that ch is an even function, that sh is an odd function.
2. Prove that for any $x \in \mathbb{R}$,

$$e^x = \text{ch}(x) + \text{sh}(x).$$

   (These two functions are the odd and even parts of the exponential function).
3. Show that for any $x \in \mathbb{R}$,

$$1 = \text{ch}^2(x) - \text{sh}^2(x).$$

Theses functions are said to be hyperbolic because of their relation with an hyperbola. Indeed, consider t to be any real number. Just as the points $(\cos(t), \sin(t))$ form a circle centerd at 0 and of radius 1, the points $(\text{ch}(t), \text{sh}(t))$ form part of the hyperbola defined by the equation $x^2 - y^2 = 1$.

Exercise 5.2 ($\star\star$)**.**

1. Analysis of the function sh.
   (a) Give the definition domain, continuity domain and differentiability domain of sh.
   (b) Compute $\lim_{x \to +\infty} \text{sh}(x)$.
   (c) Compute the derivative of sh, give the equation of its tangent at 0 and specify its position relative to the graph of sh.
   (d) Give the variation table of sh.
   (e) Draw the graph of the function.

2. Same questions with ch.
3. Same questions with tanh.

As for the cosine and sine function, many formulae exist !

Exercise 5.3 ($\star\star$)**.**
Let a and b two real numbers. Prove the following formulae :

1. $\mathrm{ch}(a+b) = \mathrm{ch}(a)\,\mathrm{ch}(b) + \mathrm{sh}(a)\,\mathrm{sh}(b)$.
2. $\mathrm{ch}(a-b) = \mathrm{ch}(a)\,\mathrm{ch}(b) - \mathrm{sh}(a)\,\mathrm{sh}(b)$.
3. $\mathrm{sh}(a+b) = \mathrm{sh}(a)\,\mathrm{ch}(b) + \mathrm{sh}(b)\,\mathrm{ch}(a)$.
4. $\mathrm{sh}(a-b) = \mathrm{sh}(a)\,\mathrm{ch}(b) - \mathrm{sh}(b)\,\mathrm{ch}(a)$.
5. $\tanh(a+b) = \frac{\tanh(a)+\tanh(b)}{1+\tanh(a)\tanh(b)}$.
6. $\tanh(a-b) = \frac{\tanh(a)-\tanh(b)}{1-\tanh(a)\tanh(b)}$.
7. Deduce the following formulae :

$$\mathrm{ch}(2a) = \mathrm{ch}^2(a) + \mathrm{sh}^2(a) = 2\,\mathrm{ch}^2(a) - 1 = 1 + \mathrm{sh}^2(a),$$

$$\mathrm{sh}(2a) = 2\,\mathrm{sh}(a)\,\mathrm{ch}(a),$$

$$\tanh(2a) = \frac{2\tanh(a)}{1+\tanh^2(a)}.$$

Exercise 5.4 ($\star\star\star$)**.**
Let g be a function defined on $\mathbb{R}$ as $g(x) = \mathrm{sh}^2(x) - 2\,\mathrm{ch}(x)$.

1. Prove that for all $x \in \mathbb{R}$, $g(x) = \mathrm{ch}^2(x) - 2\,\mathrm{ch}(x) - 1$.
2. Solve $g(x) = 0$. *(Hint : set* $\mathrm{ch}(x) = y$*)*
3. Study the variations of g. Deduce the set of real numbers x such that $g(x) \leq 0$.

Exercise 5.5 ($\star\star\star$)**.**

1. Simplify the expression

$$y = \frac{\mathrm{ch}(2\ln(x)) - \mathrm{sh}(2\ln(x))}{x}.$$

2. Solve the equation

$$5\,\mathrm{ch}(x) - 4\,\mathrm{sh}(x) = 3.$$

5.2 Study of power functions

In this section, we set :

$$\forall x \in \mathbb{R}_+^*, \forall y \in \mathbb{R}, x^y = e^{y \ln(x)}.$$

Exercise 5.6 ($\star$)**.**
Let $x \in \mathbb{R}_+^*$, $y \in \mathbb{R}_+^*$ and $(\alpha, \beta) \in \mathbb{R}^2$.
Prove the following equalities :

1. $x^{\alpha+\beta} = x^\alpha x^\beta$;
2. $(x^\alpha)^\beta = x^{\alpha\beta}$;
3. $x^{-\alpha} = \dfrac{1}{x^\alpha}$;
4. $x^{\alpha-\beta} = \dfrac{x^\alpha}{x^\beta}$;
5. $x^\alpha y^\alpha = (xy)^\alpha$;
6. $\left(\dfrac{x}{y}\right)^\alpha = \dfrac{x^\alpha}{y^\alpha}$.

Exercise 5.7 ($\star\star$)**.**
Let $a \in \mathbb{R}_+^*$ and f_a a real-valued function defined on $\mathbb{R}$ as $f_a(x) = a^x$.

1. Give $\mathcal{D}_{f_a}$, $\mathcal{D}_c$ and $\mathcal{D}_d$ respectively the definition, continuity and differentiability domains of f_a.
2. Compute for all $x \in \mathcal{D}_d$, $f_a'(x)$.
3. Compute $\lim\limits_{x \to +\infty} f_a(x)$.
4. Give, as a function of a, the variation table of f_a.
5. Draw, as a function of a, the graph of the function f_a.

Exercise 5.8 ($\star\star\star$)**.**
Let $\alpha \in \mathbb{R}$ and g_α a real-valued function defined on $\mathbb{R}$ as $g_\alpha(x) = x^\alpha$.

1. Give $\mathcal{D}_{g_\alpha}$, $\mathcal{D}_c$ and $\mathcal{D}_d$ respectively the definition, continuity and differentiability domains of g_α in the following cases :

   (a) $\alpha \in \mathbb{N}$; (b) $\alpha \in \mathbb{Z}_-^*$; (c) $\alpha \in \mathbb{R} \setminus \mathbb{Z}$.

   From now, we will only consider $\alpha \in \mathbb{R} \setminus \mathbb{Z}$.
2. Study, depending on α, the continuity and differentiability of g_α at 0.
3. Compute for any $x \in \mathcal{D}_d$, $g_\alpha'(x)$.
4. Compute $\lim\limits_{x \to +\infty} g_\alpha(x)$.
5. Give, as a function of α, the variation table of g_α.
6. Study, depending on α, the convexity (or concavity) of g_α.
7. Draw, as a function of a, the graph of the function f_a.

5.3 Study of functions

Exercise 5.9 ($\star\star$)**.**
Study the variation and draw the graph of the following functions :

1. $f_1 : x \mapsto \dfrac{x^3 - 4x}{x^2 + 3x + 2}$;
2. $f_2 : x \mapsto \exp\left(-\dfrac{2x}{x^2 - 1}\right)$;
3. $f_3 : x \mapsto \ln(\mathrm{ch}(x))$;
4. $f_4 : x \mapsto x - \ln\left(\left|\dfrac{120 + 60x + 12x^2 + x^3}{120 - 60x + 12x^2 - x^3}\right|\right)$;
5. $f_5 : x \mapsto x - 1 - \sqrt{x^2 - 1}$;
6. $f_6 : x \mapsto (1 - x)\sqrt{x^2 - 1}$;
7. $f_7 : x \mapsto \tanh\left(\dfrac{x - 1}{x + 1}\right)$;
8. $f_8 : x \mapsto \ln\left(x \ln(x)\right)$;

6 Inverse functions

Exercise 6.1 ($\star$)**.**

1. Let $f : \mathbb{R} \to \mathbb{R}$ be such that $f(x) = x^3$. Show that f is a bijection from $\mathbb{R}$ onto $\mathbb{R}$ and give its inverse function.
2. Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be such that $g(x) = x^2$. Show that g is a bijection from $\mathbb{R}_+$ onto $\mathbb{R}_+$ and give its inverse function.

Exercise 6.2 (arcsin $\star\star$)**.**
Consider the function $f : \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ such that $f(x) = \sin(x)$.

1. Show that f is bijective and denote arcsin its inverse function.
2. Compute
$$\arcsin(0), \quad \arcsin(-1), \quad \arcsin\left(\frac{\sqrt{3}}{2}\right), \quad \text{and} \quad \arcsin\left(\sin\left(\frac{15\pi}{4}\right)\right).$$
3. Give the largest sets E and F on which
$$\sin \circ \arcsin = \mathrm{Id}_E \quad \text{and} \quad \arcsin \circ \sin = \mathrm{Id}_F.$$
4. Plot the function $x \mapsto \arcsin(\sin(x))$ on $\mathbb{R}$.
5. Show that arcsin is odd, increasing and continuous.
6. Show that arcsin is differentiable on $]-1, 1[$ and that
$$\forall x \in]-1, 1[, \quad \arcsin'(x) = \frac{1}{\sqrt{1 - x^2}}.$$
(Hint : compute $\cos(\arcsin(x))$*)*
7. Plot the function arcsin.

Exercise 6.3 (arccos $\star\star$)**.**

1. Find the interval K of $\mathbb{R}$ such that the application
$$f : [0, \pi] \ni x \mapsto \cos(x) \in K$$
is bijective.
We call **Arccosine** and denote arccos the inverse function defined from K to $[0, \pi]$ as :
$$\forall (x, y) \in [0, \pi] \times K, \quad \cos(x) = y \Leftrightarrow x = \arccos(y).$$

2. Compute

$$\arccos(0), \quad \arccos(-1), \quad \arccos\left(\frac{\sqrt{3}}{2}\right), \quad \text{and} \quad \arccos\left(\cos\left(\frac{15\pi}{4}\right)\right).$$

3. Give the largest sets E and F on which

$$\cos \circ \arccos = \mathrm{Id}_E \quad \text{and} \quad \arccos \circ \cos = \mathrm{Id}_F.$$

4. Study the function $x \mapsto \arccos(\cos(x))$ on $\mathbb{R}$.
5. Show that arccos is decreasing and continuous on $[-1, 1]$.
6. Show that arccos is differentiable on $]-1, 1[$ and that

$$\forall x \in]-1, 1[, \quad \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}.$$

7. Plot the function arccos.

Exercise 6.4 (Argsh $\star\star$)**.**

1. Find the interval K of $\mathbb{R}$ such that the application

$$f : \mathbb{R} \ni x \mapsto \mathrm{sh}(x) \in K$$

is bijective.
We denote Argsh the inverse function of f.

2. Study the parity of Argsh and compute the limits of Argsh at the bounds of the interval K.
3. Study the differentiability of Argsh and compute its derivative.
4. Show that for any $x \in K$,

$$\mathrm{Argsh}(x) = \ln(x + \sqrt{x^2+1}).$$

Exercise 6.5 (Argch $\star\star$)**.**

1. Find the interval K of $\mathbb{R}$ such that the application

$$f : \mathbb{R}_+ \ni x \mapsto \mathrm{ch}(x) \in K$$

is bijective.
We denote Argch the inverse function of f.

2. Study the parity of Argch and compute the limits of Argch at the bounds of the interval K.
3. Study the differentiability of Argch and compute its derivative.
4. Compute, for any $x \in K$, the logarithmic expression of Argch.

Exercise 6.6 (Argth $\star\star$)**.**

1. Find the interval K of $\mathbb{R}$ such that the application

$$f : \mathbb{R} \ni x \mapsto \tanh(x) \in K$$

   is bijective.
   We denote Argth the inverse function of f.
2. Study the parity of Argth and compute the limits of Argth at the bounds of the interval K.
3. Study the differentiability of Argth and compute its derivative.
4. Compute, for any $x \in K$, the logarithmic expression of Argth.

7 Differentiability

Exercise 7.1 ($\star\star\star$)**.**

1. Let f be the function defined as
$$f(x) = (x^2 - 1)(x^2 - 4).$$
Without computing f', show that f' cancels out exactly three times on $\mathbb{R}$ and plot f.
2. Let $n \geq 2$ and $(p, q) \in \mathbb{R}^2$. We denote by f the polynomial function define by
$$f(x) = x^n + px + q, \quad x \in \mathbb{R}.$$
   (a) Prove that f has at most 3 reel roots.
   (b) We suppose that n is even. Show that f has at most 2 reel roots.

Exercise 7.2 ($\star\star$)**.**

Using the mean value theorem, show that

1. $\forall x \in \left(0, \frac{\pi}{2}\right]$, $1 - \cos(x) < x$.
2. $\forall x \in (0, 1)$,
$$2x < \ln\left(\frac{1+x}{1-x}\right) < \frac{2x}{1-x^2}.$$
Give upper and lower bounds of $\ln\left(\frac{5}{3}\right)$ and $\ln\left(\frac{3}{2}\right)$.

Exercise 7.3 ($\star\star\star$)**.**

1. Let f and g be two continuous functions defined on $[a, b]$ and differentiable on $]a, b[$ such that
$$\forall x \in (a, b), g'(x) \neq 0.$$
   (a) Show that $g(a) \neq g(b)$.
   (b) Show that there exists $c \in]a, b[$ such that
$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$
   (Hint : use the auxiliary function $u(x) = f(x) - kg(x)$ with a carefully chosen constant)
2. **(L'Hospital rule)** Let f and g be two continuous functions defined on an open set I, differentiable on I except maybe on a point $a \in I$.

(a) Suppose that :

$$\forall x \in I \setminus \{a\},\ g'(x) \neq 0 \quad \text{and} \quad \lim_{x \to a} \frac{f'(x)}{g'(x)} = \ell.$$

Prove that

$$\lim_{x \to a} \frac{f(x) - f(a)}{g(x) - g(a)} = \ell.$$

(b) Compute

$$\lim_{x \to \frac{\pi}{4}} \frac{2\sin^2(x) - 1}{\tan(x) - 1}.$$

3. (a) Let $a \in I$ and f be a $\mathcal{C}^1$ function on $I \setminus \{a\}$ which is continous on a. Show that if the limit of f' in a exists and it is equal to ℓ then $f \in \mathcal{C}^1(I)$ and $f'(a) = \ell$.
(Hint : apply l'Hospital rule to f and g defined by $g(x) = x$)

   (b) With the help of the following function

$$f(x) = \begin{cases} x^2 \cos\left(\frac{1}{x}\right) & \text{if} \quad x \neq 0 \\ 0 & \text{if} \quad x = 0 \end{cases},$$

show that the converse of the previous result is false.

4. Applications :

   (a) Show that $x \mapsto \begin{cases} \frac{\sin(x)}{x} & \text{if} \quad x \neq 0 \\ 1 & \text{if} \quad x = 0 \end{cases}$ is $\mathcal{C}^1$ on $\mathbb{R}$.

   (b) Let $(a, b, c) \in \mathbb{R}^3$ and f be the function defined for every x by

$$f(x) = \begin{cases} e^x & \text{if} \quad x < 0 \\ ax^2 + bx + c & \text{if} \quad x \geq 0 \end{cases}.$$

Determine for which real values a, b and c, the function f is $\mathcal{C}^1$ (resp. $\mathcal{C}^2$) on $\mathbb{R}$.

Exercise 7.4 ($\star$).

Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function such that

$$f(0) = f'(0) = f'(1) = 0 \quad \text{and} \quad f(1) = 1.$$

Defined the function $g :]0, 1[\to \mathbb{R}$ by

$$g(x) = \frac{f(x)}{x} - \frac{f(x) - 1}{x - 1}.$$

1. Show that g is continuous on $]0, 1[$.
2. Compute $\lim_{x \to 0^+} g(x)$ and $\lim_{x \to 1^-} g(x)$.
3. Show that g can be extended by continuity on $[0, 1]$.

8 Sequences of real numbers

Exercise 8.1 (Recursively defined sequences $\star$)**.**
Give the analytique expression of the sequences that follows the given recurrence equations :

1. $(u_n)_{n\in\mathbb{N}}$ given by
$$\begin{cases} u_{n+2} = 4u_{n+1} - 4u_n, & \forall n \in \mathbb{N} \\ u_0 = 1,\ u_1 = 0 \end{cases}$$
2. $(u_n)_{n\in\mathbb{N}}$ given by
$$\begin{cases} 2u_{n+2} = 3u_{n+1} - u_n, & \forall n \in \mathbb{N} \\ u_0 = 1,\ u_1 = -1 \end{cases}$$
3. $(u_n)_{n\in\mathbb{N}}$ given by
$$\begin{cases} u_{n+2} = u_{n+1} - u_n, & \forall n \in \mathbb{N} \\ u_0 = 1,\ u_1 = 2 \end{cases}$$

Exercise 8.2 (Fixed point theorems $\star\star$)**.**
Let f be a function defined on an interval I of $\mathbb{R}$. Consider a sequence defined with $u_0 \in I$ and for every $n \in \mathbb{N}$, $u_{n+1} = f(u_n)$.

1. (a) Show that if I is a stable of f, then the sequence $(u_n)_{n\in\mathbb{N}}$ is well defined.
   (b) Suppose that $I = [a, +\infty)$ with $a \in \mathbb{R}$. Show that if f is non-decreasing and a is a fixed point of f, then I is a stable set of f.
   (c) From now on, I is supposed to be a stable set of f.
      i. Show that if f is non-decreasing on I, then $(u_n)_{n\in\mathbb{N}}$ is a monotonous sequence.
      ii. Show that if f is continuous on I and if $(u_n)_{n\in\mathbb{N}}$ converges to some $\ell \in I$, then ℓ is a fixed point of f.
2. Examples :
   (a) Study the sequence defined by $u_0 \in \mathbb{R}$ and for any $n \in \mathbb{N}$, $u_{n+1} = u_n^3$.
   (b) $\forall n \in \mathbb{N}$, $u_{n+1} = \frac{1}{2}\left(u_n + \frac{a}{u_n}\right)$, with $a > 0$ and $u_0 > 0$.
   Show that the sequence $(u_n)_{n\in\mathbb{N}}$ exists and study it.

Exercise 8.3 ($\star\star$)**.**

1. Let $f : [0, 1] \to [0, 1]$ be continuous.
   (a) Prove that f admits at least on fixed point in $[0, 1]$.
   (Hint : consider the auxiliary function $u(x) = f(x) - x$)
   (b) Is this result true if f is not continuous ?
   (c) Is this result true if f is defined from $]0, 1[$ to $]0, 1[$ and continuous on $]0, 1[$?

2. Let $f, g : [0,1] \to [0,1]$ be two continuous functions such that

$$f \circ g = g \circ f.$$

The goal of this question is to prove *ad absurdio* that $f - g$ cancels at least once on $[0,1]$.
We therefore suppose that

$$\forall x \in [0,1], \quad f(x) \neq g(x).$$

   (a) Show that

$$(\forall x \in [0,1],\ f(x) < g(x)) \vee (\forall x \in [0,1],\ f(x) > g(x)).$$

   (b) Let x_0 be a fixed point of f. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence defined by induction $x_{n+1} = g(x_n)$ for all $n \in \mathbb{N}$.
      i. Show that for any $n \in \mathbb{N}$, $x_n = f(x_n)$.
      ii. Show that $(x_n)_{n\in\mathbb{N}}$ is monotonous and it converges towards a fixed point of g.

   (c) Prove that $f - g$ cancels at least once on $[0,1]$.

Exercise 8.4 (Non increasing recurrence $\star\star\star$)**.**
Let $(u_n)_{n\in\mathbb{N}}$ be the sequence defined by $u_0 \in (-\infty, -1) \cup \mathbb{R}_+^*$ and $u_{n+1} = f(u_n) = 1 + \frac{1}{u_n}$.

1. Show that the sequence $(u_n)_{n\in\mathbb{N}}$ is well defined.
2. Study the variation of f.
3. Find the fixed point of f.
4. Let $g = f \circ f$. Show that the fixed points of f are fixed points of g. Find fixed points of g and adequate stable intervals of g.
5. Let $u_0 \geq 0$. Define the sequences $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ by $\forall n \in \mathbb{N}, v_n = u_{2n}$ and $w_n = u_{2n+1}$.
   (a) Show that one of the two sequences $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ is non increasing while the other is non decreasing.
   (b) Show that $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ converge. Give their limits.
6. Study the sequence $(u_n)_{n\in\mathbb{N}}$.

Exercise 8.5 ($\star\star$)**.**
In this exercise, we use a sequence to approximate the solution of an equation. We will consider the following equation :

$$\sin(\alpha) + \frac{1}{4} = \alpha. \tag{1}$$

Let f and g define as follows :

$$\forall x \in \mathbb{R}, \quad f(x) = \sin(x) + \frac{1}{4} \quad \text{and} \quad g(x) = f(x) - x.$$

1. Existence of the solution α.
   (a) Give the variation table of g.
   (b) Prove that there exists $\alpha \in \left[0, \frac{\pi}{2}\right]$ such that $g(\alpha) = 0$.
   (c) Deduce that α is a solution of (1).
2. Approximation of α.
   We now define $(u_n)_{n\in\mathbb{N}}$ by

$$u_0 \in \left[0, \frac{\pi}{2}\right] \quad \text{and} \quad u_{n+1} = f(u_n).$$

   (a) What happens if $(u_n)_{n\in\mathbb{N}}$ converges ?
   (b) Give the variation table of f.
   (c) Prove that $[0, \alpha]$ is a stable set of f *(i.e.* $f([0, \alpha]) \subset [0, \alpha]$ *).*
   (d) Suppose that $u_0 \in [0, \alpha]$.
      i. Prove that for any $n \in \mathbb{N}$, $u_n \in [0, \alpha]$.
         (Hint : use induction)
      ii. Using the sign of g, prove that $(u_n)_{n\in\mathbb{N}}$ is non-decreasing.
      iii. Deduce that $(u_n)_{n\in\mathbb{N}}$ converges to α.
   (e) What happens if $u_0 \in \left(\alpha, \frac{\pi}{2}\right]$.

9 Spare time

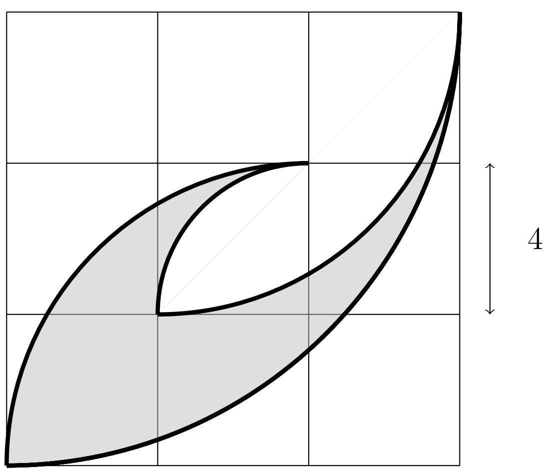
In this section, you will encounter some enjoyable mathematical problems. Most of these problems can be solved using fundamental geometric properties.

Exercise 9.1.
Let $(u_n)_{n\in\mathbb{N}^*}$ such that

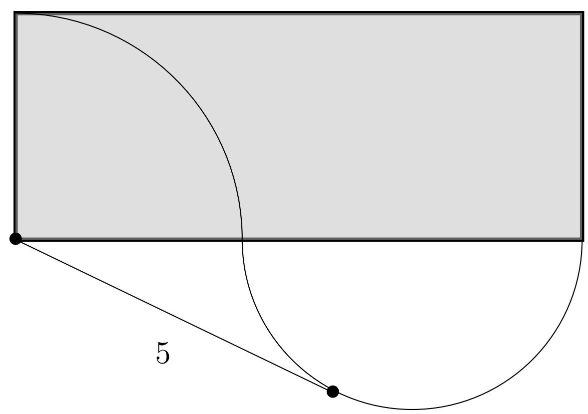
$$\begin{cases} u_{n+2} = u_{n+1} + u_n, \quad \forall n \geq 1 \\ u_2 = 2, \quad u_5 = 2024 \end{cases}$$

Find u_6.

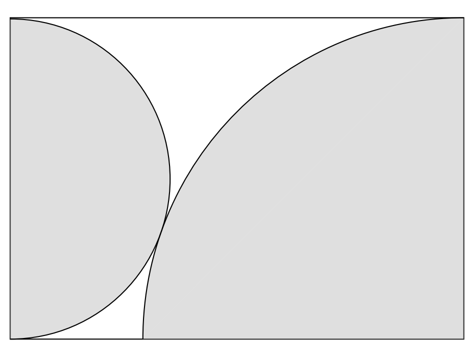
Exercise 9.2.
Find the area of the colored zone.



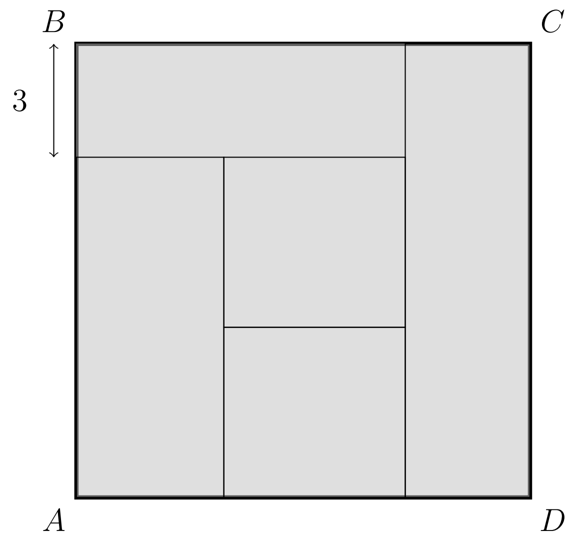
Exercise 9.3.
Find the area of the rectangle.



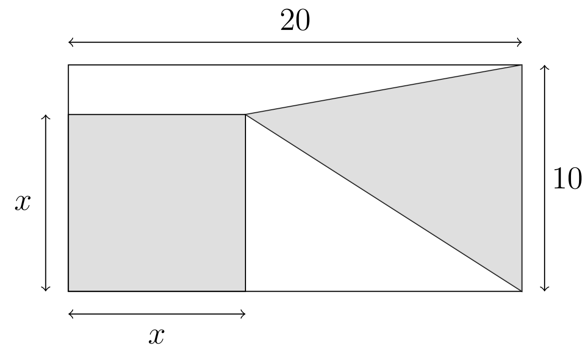
Exercise 9.4.
Find the area of the colored zone.



Exercise 9.5.
In this exercise, we suppose that the area of all the rectangles are the same. Find the area of square $ABCD$.



Exercise 9.6.
In this exercise, we suppose that the areas of the colored zones are the same. The goal is to find x.



Exercise 9.7.
Let $(x, y) \in \mathbb{R}_+$. Suppose the following

$$\begin{cases} x^2 + y^2 & = & 78 \\ xy & = & 36 \end{cases}$$

Find the value of $x^4 + y^4$.

Exercise 9.8.
Let four consecutive positive even integers such that the product of these four intergers is 13440.
What are the integers ?

Exercise 9.9.
Let $x \in \mathbb{R}_+$.

1. Solve the following equation :

$$x + \sqrt{x + \sqrt{x + \sqrt{x + \cdots}}} = 441.$$

2. Solve the following equation :

$$\sqrt{(x^x)^x} = x, \quad x > 1.$$